

HOLOMORPHIC FORMS, THE $\bar{\partial}$ -EQUATION, AND DUALITY ON A REDUCED COMPLEX SPACE

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ABSTRACT. We study two natural notions of holomorphic forms on a reduced pure n -dimensional complex space X : sections of the sheaves Ω_X^\bullet of germs of holomorphic forms on X_{reg} that have a holomorphic extension to some ambient complex manifold, and sections of the sheaves ω_X^\bullet introduced by Barlet. We show that Ω_X^p and ω_X^{n-p} are Serre dual to each other in a certain sense. We also provide explicit, intrinsic and semi-global Koppelman formulas for the $\bar{\partial}$ -equation on X and introduce fine sheaves $\mathcal{A}_X^{p,q}$ and $\mathcal{B}_X^{p,q}$ of (p,q) -currents on X , that are smooth on X_{reg} , such that $(\mathcal{A}_X^{p,\bullet}, \bar{\partial})$ is a resolution of Ω_X^p and, if Ω_X^{n-p} is Cohen-Macaulay, $(\mathcal{B}_X^{p,\bullet}, \bar{\partial})$ is a resolution of ω_X^p .

1. INTRODUCTION

In contrast to the situation on a complex manifold, on a (singular) complex space there are several different notions, serving different purposes, of holomorphic differential forms. For instance, the Grothendieck dualizing sheaf, see below, is the adequate notion of canonical sheaf when generalizing Serre duality to Cohen-Macaulay spaces. On the other hand, if one wants Kodaira-type vanishing results, then the Grauert-Riemenschneider canonical sheaf, see below, should be used. In general these sheaves of holomorphic differential forms are not equal.

From an analytic point of view there are (at least) three natural, in general different, notions of holomorphic differential forms on a reduced pure-dimensional complex space X . We will use the following terminology: The *strongly holomorphic forms*, considered, e.g., in [28], is a generalization of the strongly holomorphic functions (i.e., sections of the structure sheaf); the *weakly holomorphic forms*, studied for instance by Griffiths in [26], is a generalization of the weakly holomorphic functions and are those holomorphic forms on X_{reg} that extend to \tilde{X} , where $\tilde{X} \rightarrow X$ is a resolution of singularities; the *Barlet-Henkin-Passare holomorphic forms* are the sections of the sheaves ω_X^\bullet introduced by Barlet, [13]. These sheaves are defined in a somewhat algebraic way, but it follows from the results in [13] that the sections of ω_X^\bullet also can be described analytically as the meromorphic forms on X that are $\bar{\partial}$ -closed considered as principal value currents; in this paper we will use this analytic description as definition. Henkin and Passare emphasized and explored this analytic point of view in [28] and this is the reason for our choice of terminology.

It is clear that a strongly holomorphic form is weakly holomorphic and that a weakly holomorphic form is Barlet-Henkin-Passare holomorphic, but in general neither implication can be reversed, see, e.g., [28, Example 1]. The Grauert-Riemenschneider canonical sheaf mentioned above is the sheaf of germs of weakly holomorphic forms

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of top degree and the Grothendieck dualizing sheaf is the analogous sheaf of Barlet-Henkin-Passare holomorphic forms.

In this paper we will be concerned with strongly and Barlet-Henkin-Passare holomorphic forms and relations between these two notions. We mention however also the *reflexive differential forms*¹, defined as the reflexive hull of exterior powers of the Kähler differentials, studied, e.g., in [24] and [25] in the setting of the minimal model program, since on a normal complex space the reflexive differential forms and the Barlet-Henkin-Passare holomorphic forms are the same; this follows from [13], cf. Section 4 below. On a quasi-projective variety with at worst log terminal singularities the reflexive differential forms are also the same as the weakly holomorphic forms; this is (implied by) the main result in [25].

Let X be a reduced complex space of pure dimension n . Each point in X has a neighborhood U that can be identified with an analytic subset of some domain D in some \mathbb{C}^N ; $U \hookrightarrow D$. By pulling back the holomorphic p -forms in D to U_{reg} one obtains a notion of holomorphic p -forms on U . This construction gives us the sheaf Ω_X^p , which turns out to be an intrinsic coherent \mathcal{O}_X -module; notice in particular that $\Omega_X^0 = \mathcal{O}_X$. The sections of Ω_X^p are the strongly holomorphic p -forms.²

By the same construction one defines the sheaf $\mathcal{E}_X^{p,q}$ of smooth (p,q) -forms on X . We are interested in finding a Dolbeault-type resolution of Ω_X^p by a complex of fine sheaves, i.e., sheaves which are modules over $\mathcal{E}_X^{0,0}$. However, as opposed to the case when X is smooth, $(\mathcal{E}_X^{p,\bullet}, \bar{\partial})$ is not a resolution of Ω_X^p in general, see, e.g., [8, Example 1.1]. In order to get a tractable resolution of Ω_X^p one is thus led to consider sheaves of certain currents on X . The (p,q) -currents on X is the topological dual of the space of smooth compactly supported $(n-p, n-q)$ -forms on X . More concretely, if $i: X \hookrightarrow D \subset \mathbb{C}^N$ is an embedding and μ is a (p,q) -current on X , then $\nu := i_*\mu$ is a $(p+\kappa, q+\kappa)$ -current in D , where $\kappa := N - n$ is the codimension of X , and $\nu \cdot \xi = 0$ for any test form ξ in D whose pullback to X_{reg} vanishes. Conversely, if ν is such a current in D then there is a current μ on X such that $\nu = i_*\mu$. A current μ on X is said to have the *standard extension property* (SEP) with respect to a subvariety $Z \subset X$ if $\chi(|h|^2/\epsilon)\mu|_U \rightarrow \mu|_U$ as $\epsilon \rightarrow 0$ for all open $U \subset X$, where h is any holomorphic tuple on U that does not vanish identically on any irreducible component of $Z \cap U$ and χ is any smooth regularization of the characteristic function of $[1, \infty) \subset \mathbb{R}$; throughout the paper, χ will denote such a function. If $Z = X$ we simply say that μ has the SEP (on X). In particular, two currents with the SEP on X are equal if and only if they are equal on X_{reg} . For a pivotal example of a current with the SEP on X , let φ be a meromorphic p -form on X (possibly with values in a holomorphic line bundle $L \rightarrow X$). Then, by [30], φ defines in a canonical and robust way a principal value current on X (that takes values in L if φ does); we will usually identify a meromorphic form with its corresponding principal value current. We can now formulate our first result which is a generalization of the main result in [8] to $p \geq 1$.

¹These are sometimes also called Zariski differentials.

²Our notation is somewhat unconventional; often Ω_X^p is used to denote the Kähler-Grothendieck differential p -forms, see Section 3.

Theorem 1.1. *Let X be a reduced complex space of pure dimension n . For each $p = 0, \dots, n$ there are sheaves $\mathcal{A}_X^{p,q}$, $q = 0, \dots, n$, of (p, q) -currents on X with the SEP such that*

- (i) $\mathcal{E}_X^{p,q} \subset \mathcal{A}_X^{p,q}$ and $\oplus_q \mathcal{A}_X^{p,q}$ is a module over $\oplus_q \mathcal{E}_X^{0,q}$,
- (ii) $\mathcal{A}_{X_{reg}}^{p,q} = \mathcal{E}_{X_{reg}}^{p,q}$,
- (iii) $0 \rightarrow \Omega_X^p \hookrightarrow \mathcal{A}_X^{p,0} \xrightarrow{\bar{\partial}} \mathcal{A}_X^{p,1} \xrightarrow{\bar{\partial}} \dots \xrightarrow{\bar{\partial}} \mathcal{A}_X^{p,n} \rightarrow 0$ is an exact sheaf complex.

Since $(\mathcal{A}_X^{p,\bullet}, \bar{\partial})$ is a resolution of Ω_X^p by fine sheaves, the de Rham theorem immediately gives

Corollary 1.2. *Let X be a reduced complex space of pure dimension, let $F \rightarrow X$ be a holomorphic vector bundle, and let \mathcal{F} be the associated locally free \mathcal{O}_X -module. Then*

$$H^q(X, \mathcal{F} \otimes \Omega_X^p) \simeq H^q(\mathcal{A}^{p,\bullet}(X, F), \bar{\partial}).$$

As in [8], the construction of the \mathcal{A}_X -sheaves relies on semi-global explicit integral operators \mathcal{K} that are also used to solve the $\bar{\partial}$ -equations in Theorem 1.1 (iii). We have the following generalization of [8, Theorem 1.4].

Theorem 1.3. *Let X be a pure n -dimensional analytic subset of a pseudoconvex domain $D \subset \mathbb{C}^N$, let $D' \Subset D$ and set $X' := X \cap D'$. There are integral operators $\mathcal{K}: \mathcal{A}^{p,q}(X) \rightarrow \mathcal{A}^{p,q-1}(X')$ and $\mathcal{P}: \mathcal{A}^{p,0}(X) \rightarrow \Omega^p(X')$ such that*

$$\begin{aligned} \varphi &= \mathcal{K}(\bar{\partial}\varphi) + \mathcal{P}\varphi, \quad \varphi \in \mathcal{A}^{p,0}(X), \\ \varphi &= \bar{\partial}\mathcal{K}\varphi + \mathcal{K}(\bar{\partial}\varphi), \quad \varphi \in \mathcal{A}^{p,q}(X), \quad q \geq 1, \end{aligned}$$

as currents on X' .

This type of homotopy formulas for a Dolbeault complex are called *Koppelman formulas*.

The construction of \mathcal{P} shows that $\mathcal{P}\varphi$ has a holomorphic extension to D' . The integral operators \mathcal{K} and \mathcal{P} are given by kernels $k(\zeta, z)$ and $p(\zeta, z)$ which are currents on $X \times X'$ that are respectively integrable and smooth on $X_{reg} \times X'_{reg}$ and that have principal value-type singularities at the singular locus of $X \times X'$. In particular, one can compute $\mathcal{K}\varphi$ and $\mathcal{P}\varphi$ as

$$\mathcal{K}\varphi(z) = \lim_{\epsilon \rightarrow 0} \int_{X_\zeta} \chi_\epsilon k(\zeta, z) \wedge \varphi(\zeta), \quad \mathcal{P}\varphi(z) = \lim_{\epsilon \rightarrow 0} \int_{X_\zeta} \chi_\epsilon p(\zeta, z) \wedge \varphi(\zeta),$$

where $\chi_\epsilon := \chi(|h|^2/\epsilon)$, $h = h(\zeta)$ is a holomorphic tuple cutting out X_{sing} , and where the limit is understood in the sense of currents. Since a current locally has finite order we get the following result.

Corollary 1.4. *Let φ be a smooth $\bar{\partial}$ -closed (p, q) -form on X_{reg} such that there is a C^ℓ -smooth form in D whose pullback to X_{reg} equals φ . There is an $M_{D'} \geq 0$, independent of φ , such that the following holds.*

- (i) *If $q = 0$ and $\ell \geq M_{D'}$ then there is a $\tilde{\varphi} \in \Omega^p(X')$ such that $\varphi|_{X'_{reg}} = \tilde{\varphi}|_{X'_{reg}}$.*
- (ii) *If $q \geq 1$ and $\ell \geq M_{D'}$ then there is a smooth $(p, q-1)$ -form u on X'_{reg} such that $\bar{\partial}u = \varphi$ on X'_{reg} .*

Part (i) for $p = 0$ and $M_{D'} = \infty$ is a classical result by Malgrange [33, Théorème 4] answering a question by Grauert; for $M_{D'} < \infty$ it is due to Spallek [45]. Part (ii) for $p = 0$ and X a reduced complete intersection was first proved by Henkin and Polyakov

[29]. For $p = 0$, Corollary 1.4 is also proved in [7]. We remark that Corollary 1.4 is explicit in the sense that $\mathcal{P}\varphi$ (resp. $\mathcal{K}\varphi$) provides an explicit holomorphic extension of φ to D' (resp. explicit solution to $\bar{\partial}u = \varphi$ on X'_{reg}).

If $p + q > n$ then the equation $\bar{\partial}u = \varphi$ has a smooth solution on X_{reg} under much weaker assumptions on φ . For instance, smooth solutions on X_{reg} always exist if φ is a smooth and bounded $\bar{\partial}$ -closed (p, q) -form, $p + q > n$, on X_{reg} . If X has an isolated singularity then it is enough to assume that φ is a square-integrable smooth $\bar{\partial}$ -closed (p, q) -form, $p + q > n$, on X_{reg} . For these and other results on the $\bar{\partial}$ -equation on X_{reg} see, e.g., [14], [21], [31], [34], [35], [36], [37], [40].

The integral operators \mathcal{K} and \mathcal{P} can be applied to more general forms than sections of $\mathcal{A}_X^{p,q}$, for instance they can be applied to any semi-meromorphic form, but one has to be a bit careful with the Koppelman formulas. This is reflected in Theorem 5.4 below where we give a residue criterion for a meromorphic p -form φ to be strongly holomorphic. Moreover, if the criterion is fulfilled then $\mathcal{P}\varphi$ is a concrete holomorphic extension of φ to D' . For $p = 0$ this criterion is due to Tsikh [46] in case X is a complete intersection and to Andersson [5] in general; for $p > 0$ and X a complete intersection it is due to Henkin and Passare [28]. This residue criterion yields a geometric criterion, Proposition 5.5, which in turn leads to Proposition 1.5 below. This is a geometric characterization of complex spaces with the property that any holomorphic p -form defined on the regular part extends to a strongly holomorphic p -form. To formulate it, we need to recall the *singularity subvarieties* $S_0(\mathcal{F}) \subset S_1(\mathcal{F}) \subset \dots \subset X$ of a coherent analytic sheaf \mathcal{F} on X ; $S_\ell(\mathcal{F})$ is the set of points $x \in X$ such that $\text{depth}_{\mathcal{O}_{X,x}}(\mathcal{F}_x) \leq \ell$, cf. [44, §1]. If $i: X \rightarrow D$ is an embedding into a domain $D \subset \mathbb{C}^N$ and $(\mathcal{O}(E_\bullet, f_\bullet))$ is a locally free resolution of $\mathcal{O}(E_0)/\mathcal{I}m f_1 = i_*\mathcal{F}$, then $i(S_\ell(\mathcal{F}))$ coincides with the set $X_{N-\ell}^\mathcal{F}$ of points $x \in D$ such that $f_{N-\ell}(x)$ does not have optimal rank, cf. Section 2.3.

Proposition 1.5. *Let X be a reduced complex space of pure dimension n . Then the following conditions are equivalent.*

- (i) $\text{codim}_X X_{sing} \geq 2$ and $\text{codim}_X S_{n-k}(\Omega_X^p) \geq k + 2$ for $k \geq 1$.
- (ii) For any open $U \subset X$ the restriction map $\Omega^p(U) \rightarrow \Omega^p(U_{reg})$ is bijective.

This result is a variation on [44, Theorem 1.14], see also [43], that is explicit in the sense mentioned above. The condition (i) can be seen as a generalization to $p \geq 1$ of Serre's conditions $R1$ and $S2$ for normality. In fact, $\text{codim}_X X_{sing} \geq 2$ is $R1$ and it is known that $\text{codim}_X S_{n-k}(\mathcal{O}_X) \geq k + 2$, $k \geq 1$, is equivalent to $S2$. We mention also that if condition (i), or equivalently (ii), is satisfied for $p = n$ and one *a priori* knows that X is locally a complete intersection, then it follows that X is smooth, see Section 6.1.

We now turn our attention to Barlet-Henkin-Passare holomorphic forms. These provide an ideal framework for questions about various traces, i.e., direct images under finite holomorphic maps. This is shown in [13, Section 2] as well as in [28, Section 4] about variants of the Abel and inverse Abel theorems. Related to this is the fact that if $\text{codim}_X X_{sing} \geq 2$, then any holomorphic p -form on X_{reg} extends (necessarily uniquely) to a section of ω_X^p over X , see [13, p. 195]; cf. also Section 4 below. Moreover, if X is normal and $p \leq \text{codim}_X X_{sing} - 2$, then Flenner's result [20] shows that any holomorphic p -form on X_{reg} in fact extends to a weakly holomorphic form on X . It is also worth noticing that ω_X^p always is torsion free, see [13].

Theorem 1.6. *Let X be a reduced complex space of pure dimension n . For each $p = 0, \dots, n$ there are sheaves $\mathcal{B}_X^{p,q}$, $q = 0, \dots, n$, of (p, q) -currents on X with the SEP such that*

- (i) $\mathcal{E}_X^{p,q} \subset \mathcal{B}_X^{p,q}$ and $\oplus_q \mathcal{B}_X^{p,q}$ is a module over $\oplus_q \mathcal{E}_X^{0,q}$,
- (ii) $\mathcal{B}_{X_{reg}}^{p,q} = \mathcal{E}_{X_{reg}}^{p,q}$,
- (iii) $0 \rightarrow \mathcal{B}_X^{p,0} \xrightarrow{\bar{\partial}} \mathcal{B}_X^{p,1} \xrightarrow{\bar{\partial}} \dots \xrightarrow{\bar{\partial}} \mathcal{B}_X^{p,n} \rightarrow 0$ is a sheaf complex with coherent cohomology sheaves $\omega_X^{p,q} := \mathcal{H}^q(\mathcal{B}_X^{p,\bullet}, \bar{\partial})$ and $\omega_X^p = \omega_X^{p,0}$. If Ω_X^{n-p} is Cohen-Macaulay then $(\mathcal{B}_X^{p,\bullet}, \bar{\partial})$ is a resolution of ω_X^p .

The case $p = n$ is proved in [41]. However, the notation is not consistent, in [41] the notation $\mathcal{A}_X^{n,q}$ is used in place of $\mathcal{B}_X^{n,q}$.

The proof of Theorem 1.6 will show that if $i: X \hookrightarrow D \subset \mathbb{C}^N$, then $\omega_X^{p,q} \simeq \text{Ext}_{\mathcal{O}}^{k+q}(\Omega_X^{n-p}, \Omega^N)$, where $\mathcal{O} = \mathcal{O}_D$ and $\Omega^N = \Omega_D^N$; in this paper we will use the convention that a sheaf without a subscript is a sheaf over a suitable domain in some \mathbb{C}^N . As in [41] the \mathcal{B}_X -sheaves are defined using integral operators $\check{\mathcal{K}}$ which also fit into Koppelman-type formulas.

Theorem 1.7. *Let X be a pure n -dimensional analytic subset of a pseudoconvex domain $D \subset \mathbb{C}^N$, let $D' \Subset D$ and set $X' := X \cap D'$. There are integral operators $\check{\mathcal{K}}: \mathcal{B}^{p,q}(X) \rightarrow \mathcal{B}^{p,q-1}(X')$ and $\check{\mathcal{P}}: \mathcal{B}^{p,q}(X) \rightarrow \mathcal{B}^{p,q}(X')$ such that*

$$\psi = \bar{\partial} \check{\mathcal{K}} \psi + \check{\mathcal{K}}(\bar{\partial} \psi) + \check{\mathcal{P}} \psi$$

as currents on X' . If Ω_X^{n-p} is Cohen-Macaulay and $\psi \in \mathcal{B}^{p,q}(X)$ then $\check{\mathcal{P}} \psi \in \omega^p(X')$ if $q = 0$ and $\check{\mathcal{P}} \psi = 0$ if $q \geq 1$.

Notice that if $\psi \in \omega^p(X)$ then, on X' , $\psi = \check{\mathcal{P}} \psi$ is a representation formula for Barlet-Henkin-Passare holomorphic p -forms.

Let now temporarily X be a compact complex manifold and $F \rightarrow X$ a holomorphic vector bundle; Serre duality says that the pairing

$$(1.1) \quad H^q(\mathcal{E}^{p,\bullet}(X, F), \bar{\partial}) \times H^{n-q}(\mathcal{E}^{n-p,\bullet}(X, F^*), \bar{\partial}) \rightarrow \mathbb{C}, \quad ([\varphi]_{\bar{\partial}}, [\psi]_{\bar{\partial}}) \mapsto \int_X \varphi \wedge \psi$$

is non-degenerate. Actually, since X is assumed to be smooth, Ω_X^p is locally free (a vector bundle) and therefore it is sufficient to have (1.1) non-degenerate for $p = 0$. Via the Dolbeault isomorphism, $H^q(\mathcal{E}^{p,\bullet}(X, F), \bar{\partial}) = H^q(X, \mathcal{F} \otimes \Omega_X^p)$, where $\mathcal{F} = \mathcal{O}(F)$, we can reformulate Serre duality, for $p = 0$, algebraically as: *The pairing $H^q(X, \mathcal{F}) \times H^{n-q}(X, \mathcal{F}^* \otimes \Omega_X^n) \rightarrow \mathbb{C}$ given by the cup product is non-degenerate.* In this formulation Serre duality has been generalized to complex spaces. For instance, if X is compact and Cohen-Macaulay, then the same statement is true if we replace Ω_X^n by the Grothendieck dualizing sheaf, i.e., by ω_X^p . If X is an arbitrary compact complex space and \mathcal{F} is any coherent sheaf on X then, by the result of Ramis and Ruget, [39], there is a non-degenerate pairing $H^q(X, \mathcal{F}) \times \text{Ext}^{-q}(X; \mathcal{F}, \mathbf{K}_X^\bullet) \rightarrow \mathbb{C}$, where \mathbf{K}_X^\bullet is the *dualizing complex* in the sense of [39]. In [41], Ruppenthal, Vulcan, and the author gave a concrete analytic realization of Serre duality on reduced spaces in terms of Dolbeault cohomology completely analogous to the smooth case. We showed that one in a natural way can make sense of the current product $\varphi \wedge \psi$ for φ and ψ sections of $\mathcal{A}_X^{0,\bullet}$ and $\mathcal{B}_X^{n,\bullet}$, respectively, and moreover, if X is compact, then there is a non-degenerate pairing (1.1) with $p = 0$ if we replace $\mathcal{E}^{0,\bullet}(X, F)$ by $\mathcal{A}^{0,\bullet}(X, F)$ and $\mathcal{E}^{n,\bullet}(X, F^*)$ by $\mathcal{B}^{n,\bullet}(X, F^*)$. Since, in general, Ω_X^p is not locally free

in the singular setting it is not immediate how to generalize this to $p > 0$. We have the following result:

Theorem 1.8. *Let X be a compact reduced complex space of pure dimension n , let $F \rightarrow X$ be a holomorphic vector bundle, and let \mathcal{F} be the associated locally free sheaf. Then the pairing*

$$(1.2) \quad H^q(\mathcal{A}^{p,\bullet}(X, F), \bar{\partial}) \times H^{n-q}(\mathcal{B}^{n-p,\bullet}(X, F^*), \bar{\partial}) \rightarrow \mathbb{C},$$

$$(1.3) \quad ([\varphi]_{\bar{\partial}}, [\psi]_{\bar{\partial}}) \mapsto \int_X \varphi \wedge \psi,$$

is non-degenerate.

Notice that Theorem 1.8 connects the strongly holomorphic p -forms with the Barlet-Henkin-Passare holomorphic $n - p$ -forms. In particular, if Ω_X^p is Cohen-Macaulay, then we have a non-degenerate pairing $H^q(X, \Omega_X^p) \times H^{n-q}(X, \omega_X^{n-p}) \rightarrow \mathbb{C}$ given by the cup product.

With a slight modification of the statement, Serre duality continues to hold on paracompact spaces provided certain separability conditions are fulfilled, see, e.g., [39], and in fact, instead of proving Theorem 1.8, we will prove the following slightly more general result:

If X is a reduced paracompact complex space of pure dimension n and we replace $\mathcal{B}^{n-p,\bullet}(X, F^)$ in Theorem 1.8 by the corresponding space of sections with compact support, then the conclusion of Theorem 1.8 holds provided that $H^q(X, \mathcal{F} \otimes \Omega_X^p)$ and $H^{q+1}(X, \mathcal{F} \otimes \Omega_X^p)$ are Hausdorff.*

We remark that the Hausdorff assumption is automatically fulfilled if X is compact or holomorphically convex; if X is compact, then by the Cartan-Serre theorem the cohomology of coherent sheaves on X is finite dimensional, and if X is holomorphically convex, then by Prill's result, [38], the cohomology of coherent sheaves on X are Hausdorff. Also, by the Andreotti-Grauert theorem, $H^q(X, \mathcal{F} \otimes \Omega_X^p)$ and $H^{q+1}(X, \mathcal{F} \otimes \Omega_X^p)$ are Hausdorff for $q \geq k$ if X is k -convex.

The paper is organized as follows. In Section 2 we recall some background material that we will use. In Section 3 we show that the sheaf Ω_X^p is coherent and we introduce the notion of an $n - p$ -structure form on X ; this can be seen as a generalized Poincaré-Leray residue. In Section 4 we recall some properties of ω_X^p and show some other ones that we have not found in the literature. In Section 5 we construct intrinsic integral operators for the $\bar{\partial}$ -equation on a pure n -dimensional analytic subset of a strictly pseudoconvex domain in \mathbb{C}^N ; in Section 5.1 we also give the residue criterion for strong holomorphicity and its geometric consequence Proposition 5.5. In Section 6 we define the sheaves $\mathcal{A}_X^{\bullet,\bullet}$ and $\mathcal{B}_X^{\bullet,\bullet}$ and prove Theorem 1.1, which follows by combining Propositions 6.1 and 6.2, as well as Theorems 1.3, 1.6, and 1.7; in Section 6.1 we also prove Proposition 1.5. In Section 7 we show that Ω_X^p and ω_X^{n-p} are Serre dual to each other in the sense of Theorem 1.8.

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2. PRELIMINARIES

2.1. Meromorphic forms. Let X be a pure-dimensional analytic subset of some domain $D \subset \mathbb{C}^N$ and let W be an analytic subset containing X_{sing} but not any irreducible component of X . It is proved in [28] that the following conditions on a holomorphic p -form φ on $X \setminus W$ are equivalent. 1) φ is locally the pullback to $X \setminus W$ of a meromorphic p -form in a neighborhood of X . 2) For any desingularization $\pi: \tilde{X} \rightarrow X$ such that $\pi^{-1}X_{reg} \simeq X_{reg}$, $\pi^*\varphi$ has a meromorphic extension \tilde{X} . 3) There is a current T in D such that $i_*\varphi = T|_{D \setminus W}$, where $i: X \hookrightarrow D$ is the inclusion. 4) For any $h \in \mathcal{O}(X)$ that vanishes on W , but not identically on any component of X , the current

$$(2.1) \quad \xi \mapsto \lim_{\epsilon \rightarrow 0} \int_X \chi(|h|^2/\epsilon) \varphi \wedge \xi$$

exists and is independent of h .

The sheaf of germs of p -forms satisfying these conditions is called the sheaf of germs of *meromorphic p -forms* on X ; we will denote it by \mathcal{M}_X^p . One can check that if $x \in X$ is an irreducible point then $\mathcal{M}_{X,x}^0$ is (isomorphic to) the field of fractions of $\mathcal{O}_{X,x}$. We mention again that we usually make no distinction between a meromorphic form φ and the associated principal value current (2.1).

2.2. Pseudomeromorphic currents. Pseudomeromorphic currents were introduced in [10]; the definition we need and will use is from [8]. In one complex variable z it is elementary to see that the principal value current $1/z^m$ exists and can be defined, e.g., as the limit as $\epsilon \rightarrow 0$ in the sense of currents of $\chi(|h(z)|^2/\epsilon)/z^m$, where h is a holomorphic function (or tuple) vanishing at $z = 0$, or as the value at $\lambda = 0$ of the analytic continuation of the current-valued function $\lambda \mapsto |h(z)|^{2\lambda}/z^m$. It follows that the *residue current* $\bar{\partial}(1/z^m)$ can be computed as the limit of $\bar{\partial}\chi(|h(z)|^2/\epsilon)/z^m$ or as the value at $\lambda = 0$ of $\lambda \mapsto \bar{\partial}|h(z)|^{2\lambda}/z^m$. Since tensor products of currents are well-defined we can form the current

$$(2.2) \quad \tau = \bar{\partial} \frac{1}{z_1^{m_1}} \wedge \cdots \wedge \bar{\partial} \frac{1}{z_r^{m_r}} \wedge \frac{\gamma(z)}{z_{r+1}^{m_{r+1}} \cdots z_n^{m_n}}$$

in \mathbb{C}^n , where m_1, \dots, m_r are positive integers, m_{r+1}, \dots, m_n are nonnegative integers, and γ is a smooth compactly supported form. Notice that τ is anti-commuting in the residue factors $\bar{\partial}(1/z_j^{m_j})$ and commuting in the principal value factors $1/z_k^{m_k}$. We say that a current of the form (2.2) is an *elementary pseudomeromorphic current*. Let X be a pure-dimensional reduced complex space and let $x \in X$. We say that a germ of a current μ at x is *pseudomeromorphic* if it is a finite sum of pushforwards $\pi_*\tau = \pi_*^1 \cdots \pi_*^\ell \tau$, where \mathcal{U} is a neighborhood of x ,

$$\mathcal{U}^\ell \xrightarrow{\pi^\ell} \cdots \xrightarrow{\pi^2} \mathcal{U}^1 \xrightarrow{\pi^1} \mathcal{U}^0 = \mathcal{U},$$

each π^j is either a modification, a simple projection $\mathcal{U}^j = \mathcal{U}^{j-1} \times Z \rightarrow \mathcal{U}^{j-1}$, or an open inclusion, and τ is an elementary pseudomeromorphic current on $\mathcal{U}^\ell \subset \mathbb{C}^N$. The union of all germs of pseudomeromorphic currents on X forms an open subset of the sheaf of germs of currents on X and thus defines a subsheaf \mathcal{PM}_X . Notice that since $\bar{\partial}$ maps an elementary pseudomeromorphic current to a sum of such currents it follows that $\bar{\partial}$ maps \mathcal{PM}_X to itself.

The following result is fundamental and will be used repeatedly in this paper.

Dimension principle. *Let X be a reduced pure-dimensional complex space, let $\mu \in \mathcal{PM}(X)$, and assume that μ has support contained in a subvariety $V \subset X$. If μ has bidegree $(*, q)$ and $\text{codim}_X V > q$, then $\mu = 0$.*

This result is from [10], see also [8, Proposition 2.3]. In connection to the dimension principle we also mention that if $\mu \in \mathcal{PM}(X)$, $\text{supp } \mu \subset V$, and h is a holomorphic function vanishing on V , then $\bar{h}\mu = 0$ and $d\bar{h} \wedge \mu = 0$. Hence, if a pseudomeromorphic current μ has support contained in variety V , then there is current τ on V such that $\mu = i_*\tau$, where i is the inclusion of V , if and only if $h\mu = 0$ and $dh \wedge \mu = 0$ for all holomorphic functions h vanishing on V .

Another fundamental property of pseudomeromorphic currents is that they can be “restricted” to analytic (or constructible) subsets: Let $\mu \in \mathcal{PM}(X)$, let $V \subset X$ be an analytic subset, and set $V^c := X \setminus V$. Then the restriction of μ to the open subset V^c has a natural pseudomeromorphic extension $\mathbf{1}_{V^c}\mu$ to X . It follows that $\mathbf{1}_V\mu := \mu - \mathbf{1}_{V^c}\mu$ is a pseudomeromorphic current with support contained in V . In [10] $\mathbf{1}_{V^c}\mu$ is defined as the value at 0 of the analytic continuation of the current-valued function $\lambda \mapsto |h|^{2\lambda}\mu$, where h is any holomorphic tuple with zero set V ; $\mathbf{1}_{V^c}\mu$ can also be defined as $\lim_{\epsilon \rightarrow 0} \chi(|h|^2 v/\epsilon)\mu$, where v is any smooth strictly positive function, see [11, Lemma 3.1], cf. also [32, Lemma 6].³ Taking restrictions is commutative, in fact, if V and W are any constructible subsets then $\mathbf{1}_V\mathbf{1}_W\mu = \mathbf{1}_{V \cap W}\mu$. Let us also notice that $\mu \in \mathcal{PM}(X)$ has the SEP (on X) precisely means that $\mathbf{1}_V\mu = 0$ for all germs of analytic subsets $V \subset X$ of positive codimension. We will denote by \mathcal{W}_X the subsheaf of \mathcal{PM}_X of currents with the SEP on X . From [11, Section 3] it follows that if $\pi: X' \rightarrow X$ is either a modification, a simple projection, or an open inclusion, and $\mu \in \mathcal{W}(X')$ then $\pi_*\mu \in \mathcal{W}(X)$.

Lemma 2.1. *Let X be a reduced complex space and let $Y \subset X$ be an analytic nowhere dense subset. If $\mu \in \mathcal{PM}(X) \cap \mathcal{W}(X \setminus Y)$ then $\mathbf{1}_{X \setminus Y}\mu \in \mathcal{W}(X)$.*

Proof. Let $V \subset X$ be a germ of an analytic nowhere dense subset. Since $\mu \in \mathcal{W}(X \setminus Y)$ we see that $\text{supp } \mathbf{1}_V\mu \subset Y \cap V$ and so $\mathbf{1}_V\mathbf{1}_{X \setminus Y}\mu = \mathbf{1}_{X \setminus Y}\mathbf{1}_V\mu = 0$. \square

For future reference we give the following simple lemma, part (i) of which is almost tautological.

Lemma 2.2. *Let X be a germ of a reduced complex space and let $\mu \in \mathcal{W}(X)$.*

- (i) *We have that $\bar{\partial}\mu \in \mathcal{W}(X)$ if and only if $\lim_{\epsilon \rightarrow 0} \bar{\partial}\chi(|h|^2/\epsilon) \wedge \mu = 0$ for all generically non-vanishing holomorphic tuples h on X .*
- (ii) *Let $Y \subset X$ be an analytic nowhere dense subset, let h be a holomorphic tuple such that $Y = \{h = 0\}$, and assume that $\bar{\partial}\mu \in \mathcal{W}(X \setminus Y)$. Then $\bar{\partial}\mu \in \mathcal{W}(X)$ if and only if $\lim_{\epsilon \rightarrow 0} \bar{\partial}\chi(|h|^2/\epsilon) \wedge \mu = 0$.*

Proof. Since $\mu \in \mathcal{W}(X)$ we have that $\mu = \lim_{\epsilon \rightarrow 0} \chi(|h|^2/\epsilon)\mu$ for any generically non-vanishing h . It follows that

$$(2.3) \quad \bar{\partial}\mu = \lim_{\epsilon \rightarrow 0} \bar{\partial}(\chi(|h|^2/\epsilon)\mu) = \lim_{\epsilon \rightarrow 0} \bar{\partial}\chi(|h|^2/\epsilon) \wedge \mu + \lim_{\epsilon \rightarrow 0} \chi(|h|^2/\epsilon) \bar{\partial}\mu.$$

Now, $\bar{\partial}\mu \in \mathcal{W}(X)$ if and only if the last term on the right hand side equals $\bar{\partial}\mu$ for all generically non-vanishing h and part (i) of the lemma follows. The “only if” part of

³ ϵ -approximations and λ -approximations can be used interchangeably; λ -approximations are often computationally easier to work with while we believe that ϵ -approximations are conceptually easier. For the rest of this paper we will work with ϵ -approximations.

(ii) also follows directly from (2.3). On the other hand, if $\lim_{\epsilon \rightarrow 0} \bar{\partial}\chi(|h|^2/\epsilon) \wedge \mu = 0$ then, by (2.3), $\bar{\partial}\mu = \mathbf{1}_{X \setminus Y} \bar{\partial}\mu$ and so the “if” part of (ii) follows from Lemma 2.1. \square

Recall that a current on X is said to be semi-meromorphic if it a principal value current of the form α/f , where α is a smooth form and f is a holomorphic function or section of a complex line bundle such that f does not vanish identically on any component of X . Following [8], see also [11], we say that a current a on X is *almost semi-meromorphic* if there is a modification $\pi: X' \rightarrow X$ and a semi-meromorphic current α/f on X' such that $a = \pi_*(\alpha/f)$; if f takes values in $L \rightarrow X'$ we need also α to take values in $L \rightarrow X'$ if we want a to be scalar valued. If a is almost semi-meromorphic on X then the smallest Zariski-closed set outside of which a is smooth has positive codimension and is denoted $ZSS(a)$, the *Zariski-singular support* of a , see [11].

For proofs of the statements in this paragraph we refer to [11, Section 3], see also [8, Section 2]. Let a be an almost semi-meromorphic current on X and let $\mu \in \mathcal{PM}(X)$. Then there is a unique pseudomeromorphic current T on X coinciding with $a \wedge \mu$ outside of $ZSS(a)$ and such that $\mathbf{1}_{ZSS(a)}T = 0$. If h is a holomorphic tuple, or section of a Hermitian vector bundle, such that $\{h = 0\} = ZSS(a)$, then $T = \lim_{\epsilon \rightarrow 0} \chi(|h|^2/\epsilon)a \wedge \mu$; henceforth we will write $a \wedge \mu$ in place of T . One defines $\bar{\partial}a \wedge \mu$ so that Leibniz’ rule holds, i.e., $\bar{\partial}a \wedge \mu := \bar{\partial}(a \wedge \mu) - (-1)^{\deg a}a \wedge \bar{\partial}\mu$. If $\mu \in \mathcal{W}(X)$ then $a \wedge \mu \in \mathcal{W}(X)$; in this case $a \wedge \mu = \lim_{\epsilon \rightarrow 0} \chi(|h|^2/\epsilon)a \wedge \mu$ if h is any generically non-vanishing holomorphic section of a Hermitian vector bundle such that $\{h = 0\} \supset ZSS(a)$. If μ is almost semi-meromorphic then $a \wedge \mu$ is almost semi-meromorphic and, in fact, $a \wedge \mu = (-1)^{\deg a \deg \mu} \mu \wedge a$.

Let X be an analytic subset of pure codimension κ of some complex N -dimensional manifold D . The subsheaves of \mathcal{PM}_D of germs of $\bar{\partial}$ -closed (k, κ) -currents, $k = 0, \dots, N$, with support on X are the sheaves of Coleff-Herrera currents with support on X and are denoted \mathcal{CH}_X^k . Coleff-Herrera currents were originally introduced by Björk as the $\bar{\partial}$ -closed currents μ on D of bidegree (N, κ) such that $\bar{h}\mu = 0$ for any holomorphic function h vanishing on X and with the SEP with respect to X , see, e.g., [15]. It is proved in [4] that the definitions are equivalent. The model example is the Coleff-Herrera product: Assume that $f_1, \dots, f_\kappa \in \mathcal{O}(D)$ defines a regular sequence. Then the iteratively defined product $\bar{\partial}(1/f_1) \wedge \dots \wedge \bar{\partial}(1/f_\kappa)$ is the Coleff-Herrera product originally introduced by Coleff and Herrera in [17] in a slightly different way; cf. also [16].

Let us also notice that if X and Z are reduced pure-dimensional complex spaces and $\mu \in \mathcal{PM}(X)$, then $\mu \otimes 1 \in \mathcal{PM}(X \times Z)$, see, e.g., [8, Section 2]. We will usually omit “ $\otimes 1$ ” and simply write, e.g., $\mu(\zeta)$ to denote which coordinates μ depends on.

2.3. Residue currents associated with generically exact complexes. Let E_j , $j = 0, \dots, M$, be trivial vector bundles over an open subset of \mathbb{C}^N , let $f_j: E_j \rightarrow E_{j-1}$ be holomorphic mappings, and assume that

$$(2.4) \quad 0 \rightarrow E_M \xrightarrow{f_M} \dots \xrightarrow{f_2} E_1 \xrightarrow{f_1} E_0 \xrightarrow{f_0} 0,$$

is a complex that is pointwise exact outside of an analytic subset V of positive codimension. The bundle $E := \bigoplus_j E_j$ gets a natural superstructure by setting $E^+ := \bigoplus_j E_{2j}$ and $E^- := \bigoplus_j E_{2j+1}$. Following [9] we define currents U and R with values in

$\text{End}(E)$ associated with (2.6) and a choice of Hermitian metrics on the E_k .⁴ Notice that $\text{End}(E)$ gets an induced superstructure and so spaces of forms and currents with values in E or $\text{End}(E)$ get superstructures as well. Let $f := \oplus_j f_j$ and set $\nabla := f - \bar{\partial}$, which then becomes an odd mapping on spaces of forms or currents with values in E such that $\nabla^2 = 0$; notice that ∇ induces an odd mapping ∇_{End} on $\text{End}(E)$ -valued forms or currents such that $\nabla_{\text{End}}^2 = 0$. Outside of V , let $\sigma_k: E_{k-1} \rightarrow E_k$ be the pointwise minimal inverse of f_k , i.e., for each $z \notin V$,

$$\sigma_k(z)f_k(z) = \Pi_{(\text{Ker } f_k(z))^\perp}, \quad f_k(z)\sigma_k(z) = \Pi_{\text{Im } f_k(z)},$$

where Π denotes orthogonal projection. Set $\sigma := \sigma_1 + \sigma_2 + \dots$ and let $u := \sigma + \sigma\bar{\partial}\sigma + \sigma(\bar{\partial}\sigma)^2 + \dots$. Notice that

$$u = \sum_{0 \leq k < \ell} u_\ell^k,$$

where $u_\ell^k := \sigma_\ell \bar{\partial} \sigma_{\ell-1} \cdots \bar{\partial} \sigma_{k+1}$ is a smooth $\text{Hom}(E_k, E_\ell)$ -valued $(0, \ell - k - 1)$ -form outside of V . One can show that $\nabla_{\text{End}} u = \text{Id}_E$. We extend u as a current across V by setting

$$U := \lim_{\epsilon \rightarrow 0} \chi(|F|^2/\epsilon)u,$$

where F is a (non-trivial) holomorphic tuple vanishing on V , cf., [9, Section 2] and [2, Theorem 5.1]. As with u we will write $U = \sum_{0 \leq k < \ell} U_\ell^k$, where now U_ℓ^k is a $\text{Hom}(E_k, E_\ell)$ -valued $(0, \ell - k - 1)$ -current.

Remark 2.3. The procedure of taking pointwise minimal inverses produce almost semi-meromorphic currents, see, e.g., [11, Section 4]. Thus the σ_j have almost semi-meromorphic extensions across V and, letting σ_j denote the extension as well, we have $U_\ell^k := \sigma_\ell \bar{\partial} \sigma_{\ell-1} \cdots \bar{\partial} \sigma_{k+1}$, where the products are in the sense of Section 2.2 above. In particular, each U_ℓ^k is an almost semi-meromorphic current in (some domain in) \mathbb{C}^N .

The current R is defined by

$$\nabla_{\text{End}} U = \text{Id}_E - R$$

and hence R is supported on V and $\nabla_{\text{End}} R = 0$. Notice that R is an almost semi-meromorphic current plus $\bar{\partial}$ of such a current. One can check that

$$(2.5) \quad R = \lim_{\epsilon \rightarrow 0} (1 - \chi(|F|^2/\epsilon)) \text{Id}_E + \bar{\partial} \chi(|F|^2/\epsilon) \wedge u.$$

We write $R = \sum_{0 \leq k < \ell} R_\ell^k$, where R_ℓ^k is a $\text{Hom}(E_k, E_\ell)$ -valued $(0, \ell - k)$ -current.

Now consider the sheaf complex

$$(2.6) \quad 0 \rightarrow \mathcal{O}(E_M) \xrightarrow{f_M} \cdots \xrightarrow{f_2} \mathcal{O}(E_1) \xrightarrow{f_1} \mathcal{O}(E_0)$$

associated with (2.6) and set $\mathcal{F} = \mathcal{O}(E_0)/\mathcal{I}_{\mathcal{M}} f_1$. Let Z_k be the set where f_k does not have optimal rank; it is well-known that Z_k is analytic. By the Buchsbaum-Eisenbud criterion the complex (2.6) is exact, and thus a free resolution of \mathcal{F} , if and only if $\text{codim } Z_k \geq k$, $k \geq 1$. In this case the Z_k are invariants of \mathcal{F} . Moreover, by [19, Corollary 20.14], $\text{codim } Z_k \geq k + 1$ for $k \geq \kappa + 1$ if and only if \mathcal{F} has pure dimension, i.e., no stalk \mathcal{F}_x has embedded primes. By, e.g., [19, Corollary 20.12],

$$\cdots \subset Z_k \subset Z_{k-1} \subset \cdots \subset Z_{\kappa+1} \subsetneq Z_\kappa = \cdots = Z_1,$$

⁴That a current takes values in a vector bundle F means that it acts on test-forms with values in F^* .

where κ is the codimension of \mathcal{F} (i.e., the largest integer such that the zero sets of the associated primes of each stalk \mathcal{F}_x have codimension $\geq \kappa$) and Z_1 is the zero set of the ideal sheaf $\text{Ann } \mathcal{F}$. In particular notice that $\mathcal{F}_x = 0$ for $x \notin Z_1$. It is straightforward to check that Z_k is the set of points $x \in \mathbb{C}^N$ such that the projective dimension of \mathcal{F}_x is $\geq k$. From the Auslander-Buchsbaum formula it thus follows that $S_\ell(\mathcal{F}|_{Z_1}) = Z_{N-\ell}$. We recall also that \mathcal{F} is Cohen-Macaulay if and only if $Z_k = \emptyset$ for $k \geq \kappa + 1$, i.e., if and only if there is a resolution (2.6) of \mathcal{F} with $M = \kappa$.

It is proved in [9] that if (2.6) is exact then $R = \sum_{\ell \geq \kappa} R_\ell^0$ and moreover, a section φ of $\mathcal{O}(E_0)$ is in $\mathcal{Im} f_1$ if and only if (the E -valued) current $R\varphi$ vanishes.

Example 2.4. The model example is the Koszul complex: Let $f_1, \dots, f_\kappa \in \mathcal{O}(D)$ (D a domain in \mathbb{C}^N) be a regular sequence and let (2.6) with $M = \kappa$ be the associated Koszul complex, which then is a resolution of $\mathcal{O}/\langle f_1, \dots, f_\kappa \rangle$. With the trivial metric on the bundles E_j the resulting R is $R_{BM} \wedge e_\kappa \wedge e_0^*$, where R_{BM} the residue current of Bochner-Martinelli type introduced in [42] and e_0 and e_κ are suitable frames for the line bundles E_0 and E_κ respectively. It is shown in [42], see also [4], that R_{BM} equals the Coleff-Herrera product in the present situation. By [9, Theorem 4.1], R is in fact independent of the choice of Hermitian metric and so the above procedure always produce the Coleff-Herrera product (times $e_\kappa \wedge e_0^*$) in the case of regular sequences.

3. STRONGLY HOLOMORPHIC p -FORMS ON X

Let $X = \{\tilde{f}_1 = \dots = \tilde{f}_r = 0\}$ be a pure n -dimensional analytic subset of a neighborhood of 0 in \mathbb{C}^N and set $\kappa := N - n$; assume that $0 \in X$. Let $\{\varphi_j\}$ and $\{\psi_j\}$ be finite sets of generators for Ω^p and Ω^{p-1} respectively and let $\tilde{\mathcal{J}}_X^p \subset \Omega^p$ be the coherent subsheaf generated over \mathcal{O} by $\{\tilde{f}_i \varphi_j\}$ and $\{d\tilde{f}_i \wedge \psi_j\}$. It is clear that $\tilde{\mathcal{J}}_{X,x}^p = \Omega_x^p$ for x outside of X , that $\text{codim } \Omega_x^p/\tilde{\mathcal{J}}_{X,x}^p = \kappa$, and that $\Omega_x^p/\tilde{\mathcal{J}}_{X,x}^p \simeq \Omega_{X,x}^p$ if $x \in X_{reg}$. The sections of $\Omega^p/\tilde{\mathcal{J}}_X^p$ are the Kähler-Grothendieck differential p -forms on X ; the classical Kähler differentials correspond to $\Omega^1/\tilde{\mathcal{J}}_X^1$. In general, $\Omega^p/\tilde{\mathcal{J}}_X^p$ has torsion and is not of pure dimension.

Example 3.1. If $X = \{z_1^2 = z_2^3\} \subset \mathbb{C}^3$ then $\varphi = 2z_2 dz_1 - 3z_1 dz_2$ is not in $\tilde{\mathcal{J}}_{X,0}^1$, and thus defines a non-zero Kähler differential, even though the pullback of φ to X_{reg} vanishes. More generally, if X is a germ of an arbitrary reduced planar singular curve at $0 \in \mathbb{C}^2$, then, as one can check, $\Omega_0^1/\tilde{\mathcal{J}}_{X,0}^1$ always has embedded primes.

From a primary decomposition of $\tilde{\mathcal{J}}_{X,0}^p$ we see that there are coherent sheaves \mathcal{J}_X^p and \mathcal{S}_X^p in a neighborhood $\mathcal{U} \subset D$ of 0 such that

$$\tilde{\mathcal{J}}_X^p = \mathcal{J}_X^p \cap \mathcal{S}_X^p,$$

Ω^p/\mathcal{J}_X^p has pure codimension κ , and Ω^p/\mathcal{S}_X^p has codimension $> \kappa$. It follows that $\tilde{\mathcal{J}}_X^p = \mathcal{J}_X^p$ outside of an analytic set of codimension $> \kappa$. Hence, $\tilde{\mathcal{J}}_X^p = \mathcal{J}_X^p$ generically on X and so, the pullback of any section of \mathcal{J}_X^p to X_{reg} vanishes. On the other hand, since Ω^p/\mathcal{J}_X^p has pure codimension it follows that any section φ of Ω^p such that the pullback of φ to X_{reg} vanishes in fact is a section of \mathcal{J}_X^p ; this is well known and also follows from Proposition 3.4 below. Hence, \mathcal{J}_X^p is the sheaf of germs of holomorphic p -forms φ in \mathcal{U} such that $\varphi \wedge [X] = 0$ and Ω^p/\mathcal{J}_X^p is the sheaf of germs of strongly holomorphic p -forms on $X \cap \mathcal{U}$; in particular, Ω_X^p is coherent.

Remark 3.2. The reader familiar with gap-sheaves will recognize \mathcal{J}_X^p as the *relative gap-sheaf of $\tilde{\mathcal{J}}_X^p$ in Ω^p with respect to X_{sing}* ; see, e.g., [44, p. 47].

For simplicity we will for the rest of this section assume that X and \mathcal{J}_X^p are defined in a neighborhood of the closure of the unit ball \mathbb{B} of \mathbb{C}^N and we denote the inclusion $X \hookrightarrow \mathbb{B}$ by i . Moreover, we let (2.6) be a resolution of $i_*\Omega_X^p = \Omega^p/\mathcal{J}_X^p$ with $E_0 = \Lambda^{p,0}T^*\mathbb{C}^N$ so that $\mathcal{O}(E_0) = \Omega^p$; recall also the associated sets Z_k , cf. Section 2.3. Since Ω_X^p has pure codimension we have $\text{codim } Z_k \geq k+1$, for $k = \kappa+1, \kappa+2, \dots$, and in particular $Z_N = \emptyset$. Hence, we can, and will, assume that $M \leq N-1$ in (2.6). The resolution (2.6) induces a complex (2.4) that is pointwise exact outside of X . A choice of Hermitian metrics on the E_j gives us associated $\text{Hom}(E_0, E)$ -valued currents U and R so that, in particular, a holomorphic p -form φ is a section of \mathcal{J}_X^p if and only if the E -valued current $R\varphi$ vanishes.

Example 3.3. Assume that $X = \{w_1 = \dots = w_\kappa = 0\}$, where $(z_1, \dots, z_n; w_1, \dots, w_\kappa)$ are local coordinates in an open subset U of \mathbb{C}^N . A basis for the $(p, 0)$ -forms in U is given by the union of $\{dz_I \wedge dw_J\}$, where I and J range over increasing multiindices such that $|I| + |J| = p$. Let E'_0 and E''_0 be the subbundles of $\Lambda^{p,0}T^*U$ generated by dz_I , $|I| = p$, and $dz_J \wedge dw_K$, $|J| < p$, respectively. It is clear that \mathcal{J}_X^p is generated by $w_i dz_J$, $i = 1, \dots, \kappa$, $|J| = p$ and $dz_I \wedge dw_J$, $|J| \geq 1$. To get a resolution of Ω_X^p we let, for each increasing multiindex $J \subset \{1, \dots, n\}$ with $|J| = p$, $(E_\bullet^J, f_\bullet^J)$ be the Koszul complex corresponding to w_1, \dots, w_κ and we identify E_0^J with the line bundle generated by dz_J ; notice that $\bigoplus_{|J|=p} E_0^J = E'_0$. It is well-known that $(\mathcal{O}(E_\bullet^J), f_\bullet^J)$ is a resolution of the quotient $\mathcal{O}dz_J/(w_1, \dots, w_\kappa)\mathcal{O}dz_J$. Let (E'_\bullet, f'_\bullet) be the direct sum of the complexes $(E_\bullet^J, f_\bullet^J)$ over all increasing multiindices J with $|J| = p$. Then

$$(3.1) \quad 0 \rightarrow \mathcal{O}(E'_\kappa) \xrightarrow{f'_\kappa} \dots \xrightarrow{f'_3} \mathcal{O}(E'_2) \xrightarrow{f'_2} \mathcal{O}(E'_1) \oplus \mathcal{O}(E''_0) \xrightarrow{f'_1 \oplus \text{Id}} \mathcal{O}(E'_0) \oplus \mathcal{O}(E''_0)$$

is a resolution of Ω_X^p since (3.1) is exact (as a direct sum of exact complexes) and the cokernel of the map $f'_1 \oplus \text{Id}$ equals Ω_X^p .

Since w_1, \dots, w_κ is a regular sequence it follows that, for any choice of Hermitian metrics on the E_i^J , the current R^J associated with $(E_\bullet^J, f_\bullet^J)$ equals

$$R^J = \varepsilon^J \otimes (dz_J)^* \otimes \bar{\partial} \frac{1}{w_1} \wedge \dots \wedge \bar{\partial} \frac{1}{w_\kappa},$$

where ε^J is a frame for E_κ^J , $(dz_J)^*$ is the dual of dz_I , and $\bar{\partial}(1/w_1) \wedge \dots \wedge \bar{\partial}(1/w_\kappa)$ is the Coleff-Herrera product, cf. Example 2.4. Choosing a metric that respects the direct sum structure we get that the current R associated with (3.1) equals

$$R = \sum'_{|J|=p} \varepsilon^J \otimes (dz_J)^* \otimes \bar{\partial} \frac{1}{w_1} \wedge \dots \wedge \bar{\partial} \frac{1}{w_\kappa}.$$

Notice that by the Poincare-Lelong formula

$$R \wedge dw \wedge dz = \pm(2\pi i)^\kappa \sum'_{|J|=p} \varepsilon^J \otimes (dz_J)^* \otimes [X] \wedge dz.$$

Moreover, if φ is a (p, n) -test form in U then we can also view φ as an E_0 -valued $(0, n)$ -test form and we see that

$$R \wedge dw \wedge dz \cdot \varphi = \pm (2\pi i)^\kappa \sum'_{|J|=p} \varepsilon^J \int_X dz_{J^c} \wedge \varphi,$$

where $J^c = \{1, \dots, n\} \setminus J$.

The preceding example indicates that the $(N, *)$ -current $R \wedge dz$ is of the form $i_* \mu$ for some $(n-p, *)$ -current μ on $i: X \hookrightarrow \mathbb{B}$; here and in the rest of the paper, $dz := dz_1 \wedge \dots \wedge dz_N$. At first sight this seems to contradict the discussion in Paragraph 2 of the introduction. To shed some light on this notice first that $R_k \wedge dz$ is a (distribution-valued) section of $E_k \otimes E_0^* \otimes \Lambda^{N,k} T^* \mathbb{B}$ and recall that $E_0 = \Lambda^{p,0} T^* \mathbb{B}$. Interior multiplication induces a natural isomorphism $E_0^* \otimes \Lambda^{N,k} T^* \mathbb{B} \rightarrow \Lambda^{N-p,k} T^* \mathbb{B}$ and moreover, if φ is an E_0 -valued $(0, N-k)$ -form then we can also view it as a $(p, N-k)$ -form $\tilde{\varphi}$. We get a diagram

$$\begin{array}{ccc} E_0^* \otimes \Lambda^{N,k} T^* \mathbb{B} & \xrightarrow{\varphi} & \Lambda^{N,N} T^* \mathbb{B} \\ \downarrow & & \parallel \\ \Lambda^{N-p,k} T^* \mathbb{B} & \xrightarrow{\wedge \tilde{\varphi}} & \Lambda^{N,N} T^* \mathbb{B} \end{array}$$

where φ also denotes the natural map induced by φ , and the map $\wedge \tilde{\varphi}$ is defined by taking wedge product with $\tilde{\varphi}$. The diagram commutes, as can be checked, and therefore we can view $R \wedge dz$ either as an $(N, *)$ -current with values in $\text{Hom}(E_0, E) \simeq E \otimes E_0^*$ or as an $(N-p, *)$ -current with values in E ; with the first viewpoint $R \wedge dz$ acts naturally on $E_0 \otimes E^*$ -valued $(0, *)$ -test forms and with the second one it acts on E^* -valued $(p, *)$ -test forms and the result is the same. For future reference we also note that with the first point of view $R \wedge dz$ can be naturally multiplied with smooth E_0 -valued $(0, *)$ -forms yielding E -valued currents; with the second point of view $R \wedge dz$ can be naturally multiplied with scalar-valued $(p, *)$ -forms yielding the same E -valued currents. Unless explicitly said, we will use the second point of view (even though the notation might suggest otherwise).

Proposition 3.4. *There is a unique almost semi-meromorphic current $\omega = \omega_0 + \omega_1 + \dots + \omega_{n-1}$ on X , where ω_k is an $E_{\kappa+k}$ -valued $(n-p, k)$ -current, such that*

$$R \wedge dz = i_* \omega.$$

The current ω has the following additional structure.

- (i) *If Ω_X^p is Cohen-Macaulay, then ω_0 is an E_κ -valued section of ω_X^{n-p} over X . In general, there is a $\tilde{\omega}_0 \in \omega^{n-p}(X, V)$, where V is an auxiliary trivial vector bundle over \mathbb{B} , and an almost semi-meromorphic $\text{Hom}(V, E_\kappa)$ -valued $(0, 0)$ -current α_0 in \mathbb{B} that is smooth outside of $Z_{\kappa+1}$ such that $\omega_0 = \alpha_0|_X \tilde{\omega}_0$ as currents.*
- (ii) *For $k \geq 1$ there are almost semi-meromorphic $(0, 1)$ -currents α_k in \mathbb{B} with values in $\text{Hom}(E_{\kappa+k-1}, E_{\kappa+k})$ that are smooth outside of $Z_{\kappa+k}$ and such that $\omega_k = \alpha_k|_X \omega_{k-1}$ as currents.*

This proposition is the analogue of [8, Proposition 3.3] and the proof is essentially the same. However, because of the fundamental importance of Proposition 3.4 for this paper we include the proof.

The form ω will be called an $n-p$ -structure form.

Proof. It is well-known [23, p. 72] that one can choose $\tilde{f}_1, \dots, \tilde{f}_\kappa \in \mathcal{J}_X^0 \subset \mathcal{O}$ that define a complete intersection $\tilde{X} = \{\tilde{f}_1 = \dots = \tilde{f}_\kappa = 0\}$ such that X is a union of irreducible components of \tilde{X} and $d\tilde{f}_1 \wedge \dots \wedge d\tilde{f}_\kappa$ is generically non-vanishing on each component of X . Let $z = (z'; z'') = (z'_1, \dots, z'_n; z''_1, \dots, z''_\kappa)$ be coordinates in \mathbb{C}^N and let $w_j := \tilde{f}_j(z)$, $j = 1, \dots, \kappa$; then $(z'; w_1, \dots, w_\kappa)$ are local coordinates in a neighborhood of any point in the open set U where $h := \det(\partial \tilde{f}_i / \partial z''_j)_{i,j=1}^\kappa$ is non-vanishing.

Let W be a hypersurface that intersects X properly and such that $X \setminus W \subset X \cap U \subset X_{reg}$ and $\tilde{\mathcal{J}}_{X,x}^p = \mathcal{J}_{X,x}^p$ for $x \in X \setminus W$; let $x \in X \setminus W$. In a neighborhood of x the Koszul-type complex (3.1) is a minimal resolution of Ω_X^p and therefore the resolution (2.6) (restricted to a neighborhood of x) contains (3.1) as a direct summand. By [9, Theorem 4.4] we get that

$$R_\kappa = \alpha \otimes \bar{\partial} \frac{1}{w_1} \wedge \dots \wedge \bar{\partial} \frac{1}{w_\kappa},$$

where α is a smooth section of $\text{Hom}(E_0, E_\kappa)$ in a neighborhood of x . Since $dz' \wedge dw = h dz' \wedge dz'' = h dz$ follows from the Poincaré-Lelong formula that

$$R_\kappa \wedge dz = \pm(2\pi i)^\kappa \alpha \otimes \frac{1}{h} [X] \wedge dz'$$

in a neighborhood of x ; here we view $R_\kappa \wedge dz$ as a $\text{Hom}(E_0, E_\kappa)$ -valued (N, κ) -current. In view of the discussion preceding the proposition we may view $\pm(2\pi i)^\kappa \alpha \otimes dz'/h$ as an E_κ -valued $(n-p, 0)$ -form $\tilde{\alpha}$. Clearly $\tilde{\alpha}$ is uniquely determined by $R_\kappa \wedge dz$ and so all local $\tilde{\alpha}$ glue together to a smooth E_κ -valued $(n-p, 0)$ -form ω_0 on $X \setminus W$ and

$$(3.2) \quad R_\kappa \wedge dz = \omega_0 \wedge [X] \quad \text{in } \mathbb{B} \setminus W.$$

If Ω_X^p is Cohen-Macaulay, then we can choose a resolution (2.6) with $M = \kappa$. Hence, $R = R_\kappa$ is $\bar{\partial}$ closed and thus ω_0 is holomorphic on $X \setminus W$. Since $R_\kappa \wedge dz$ is a current extension of $\omega_0 \wedge [X]$ from $\mathbb{B} \setminus W$ to \mathbb{B} , it follows from [28, Theorem 1] that there is a meromorphic form in a neighborhood of X whose pullback to $X \setminus W$ equals ω_0 . Then, since $R_\kappa \wedge dz$ in addition is $\bar{\partial}$ -closed it follows from [28, Proposition 1] that ω_0 extends to a section, still denoted ω_0 , of ω_X^{n-p} over X . Since, by the dimension principle, R_κ has the SEP with respect to X it follows that $R_\kappa \wedge dz = i_* \omega_0$ in \mathbb{B} .

In general, the sheaf $\mathcal{H}\text{er}(f_{\kappa+1}^* : \mathcal{O}(E_\kappa^*) \rightarrow \mathcal{O}(E_{\kappa+1}^*))$ is at least coherent and so there is a trivial vector bundle V and a holomorphic morphism $g : \mathcal{O}(E_\kappa) \rightarrow \mathcal{O}(V)$ such that

$$(3.3) \quad \mathcal{O}(V^*) \xrightarrow{g^*} \mathcal{O}(E_\kappa^*) \xrightarrow{f_{\kappa+1}^*} \mathcal{O}(E_{\kappa+1}^*)$$

is exact. Notice that outside of $Z_{\kappa+1}$, $f_{\kappa+1}$ has optimal rank and so $f_{\kappa+1}^*$ has as well. Then, since (3.3) is exact it follows that

$$(3.4) \quad E_{\kappa+1,x} \xrightarrow{f_{\kappa+1}(x)} E_{\kappa,x} \xrightarrow{g(x)} V_x, \quad x \notin Z_{\kappa+1}, \quad \text{is exact.}$$

In particular, $gf_{\kappa+1} = 0$ outside of X and so, in view of (2.5), we get

$$\bar{\partial} g R_\kappa = g \bar{\partial} R_\kappa = g f_{\kappa+1} R_{\kappa+1} = 0.$$

Thus by (3.2) $g\omega_0$ is holomorphic in $X \setminus W$ and as above it follows from [28] that $g\omega_0$ extends to a V -valued section $\tilde{\omega}_0$ of ω_X^{n-p} over X . Moreover, from (3.4), the fact that (2.4) is exact in $\mathbb{B} \setminus X$, and the definition of σ_κ it follows that

$$(3.5) \quad (\text{Ker } g)^\perp = (\text{Im } f_{\kappa+1}(x))^\perp = (\text{Ker } f_\kappa(x))^\perp = \text{Im } \sigma_\kappa(x), \quad x \in \mathbb{B} \setminus X.$$

Now for $x \in \mathbb{B} \setminus Z_{\kappa+1}$ we let $\alpha_0(x): V_x \rightarrow E_{\kappa,x}$ be the minimal inverse of $g(x)$; notice in particular that $\alpha_0 g = \text{Id}_{(\text{Ker } g)^\perp}$ in $\mathbb{B} \setminus X$. It follows, cf. Remark 2.3, that α_0 has an almost semi-meromorphic extension, still denoted α_0 , across $Z_{\kappa+1}$. Recall that $u_\kappa^0 = \sigma_\kappa \bar{\partial} \sigma_{\kappa-1} \cdots \bar{\partial} \sigma_1$ so that u_κ^0 takes values in $\text{Hom}(E_0, \text{Im } \sigma_\kappa)$ where it is defined, i.e., in $\mathbb{B} \setminus X$. In view of (2.5) and (3.5) we get

$$(3.6) \quad \alpha_0 g R_\kappa = R_\kappa$$

in $\mathbb{B} \setminus Z_{\kappa+1}$ since α_0 is smooth there; the dimension principle then shows that (3.6) holds in \mathbb{B} . From (3.2) we then see that $\omega_0 = \alpha_0 g \omega_0 = \alpha_0 \tilde{\omega}_0$ in $X \setminus W$ and so $\alpha_0 \tilde{\omega}_0$ is an extension of ω_0 to X of the desired form. But $i_* \alpha_0 \tilde{\omega}_0$ and $R_\kappa \wedge dz$ both have the SEP with respect to X and so it follows from (3.2) that

$$(3.7) \quad R_\kappa \wedge dz = i_* \alpha_0 \tilde{\omega}_0, \quad \text{in } \mathbb{B}.$$

From the proof of [9, Theorem 4.4], see also Remark 2.3, we know that, for $k \geq 1$ there are $\text{Hom}(E_{\kappa+k-1}, E_{\kappa+k})$ -valued almost semi-meromorphic $(0, 1)$ -currents α_k in \mathbb{B} that are smooth outside of $Z_{\kappa+k}$ and such that

$$(3.8) \quad R_{\kappa+k} = \alpha_k R_{\kappa+k-1}, \quad k \geq 1,$$

in $\mathbb{B} \setminus Z_{\kappa+k}$. However, since $\text{codim } Z_{\kappa+k} \geq \kappa+k+1$ for $k \geq 1$ the dimension principle show that (3.8) holds in \mathbb{B} . If we inductively define $\omega_k := \alpha_k \omega_{k-1}$ for $k \geq 1$ it follows from (3.7) and (3.8) that $R_{\kappa+k} \wedge dz = i_* \alpha_k \tilde{\omega}_{k-1}$ in \mathbb{B} . \square

Since $R \wedge dz = i_* \omega$, where ω is almost semi-meromorphic on X , it follows that R has the SEP with respect to X . In particular, if φ is a holomorphic p -form in ambient space such that the pullback of φ to X_{reg} vanishes, then the (E -valued) current $R\varphi$ vanishes, i.e., φ is a section of \mathcal{J}_X^p .

Lemma 3.5. *If φ is a smooth $(n-p, q)$ -form on X then there is a smooth $(0, q)$ -form ϕ on X with values in $E_\kappa^*|_X$ such that $\varphi = \omega_0 \wedge \phi$.*

Proof. Consider a smooth extension of φ to \mathbb{B} ; it can be written in the form $\sum_j \varphi'_j \wedge \varphi''_j$ where φ'_j is a holomorphic $n-p$ -form in \mathbb{B} and φ''_j is a smooth $(0, q)$ -form in \mathbb{B} . The $(N-p, \kappa)$ -current $\varphi'_j \wedge [X]$ can be viewed as a section of $\mathcal{H}\text{om}_{\mathcal{O}}(\Omega_X^p, \mathcal{CH}_X^N)$ via $\Omega_X^p \ni \psi \mapsto \psi \wedge \varphi'_j \wedge [X]$. By Proposition 4.1 below (with p and $n-p$ interchanged) there is a section ξ_j of $\mathcal{O}(E_\kappa^*)$ such that $i_*(i^* \xi_j \cdot \omega_0) = \varphi'_j \wedge [X]$. It follows that $\varphi = \sum_j i^* \xi_j \cdot \omega_0 \wedge i^* \varphi''_j$. \square

4. BARLET-HENKIN-PASSARE HOLOMORPHIC p -FORMS

The sheaf ω_X^p was introduced by Barlet in [13] as the kernel of a natural map $j_* j^* \Omega_X^p \rightarrow \mathcal{H}_{X_{\text{sing}}}^1(\mathcal{E}\text{xt}_{\mathcal{O}}^{\kappa}(\mathcal{O}_X, \Omega^{N-p}))$, where $j: X_{\text{reg}} \hookrightarrow X$ is the inclusion, and it is proved, [13, Proposition 4], that the sections of ω_X^p can be identified with the holomorphic p -forms on X_{reg} that have an extension to X as a $\bar{\partial}$ -closed current without any mass concentrated on X_{sing} . Moreover, it is shown that ω_X^p is coherent and so ω_X^p / Ω_X^p is a coherent sheaf supported on X_{sing} . Hence, locally, for a suitable generically non-vanishing holomorphic function h , one has $h \omega_X^p \subset \Omega_X^p$ and it follows that ω_X^p indeed can be identified with the sheaf of germs of meromorphic p -forms on X that are $\bar{\partial}$ -closed considered as principal value currents.

Let X be a pure n -dimensional analytic subset of a neighborhood of $\overline{\mathbb{B}} \subset \mathbb{C}^N$, set $\kappa = N-n$, and let (2.6) be a resolution of $\Omega_X^{n-p} = \Omega^{n-p} / \mathcal{J}_X^{n-p}$ in \mathbb{B} ; notice that now $\mathcal{O}(E_0) = \Omega^{n-p}$. Let $R = R_\kappa + \dots$ be the current associated with (2.6) (for some

choice of Hermitian metrics), let $i_*\omega = R \wedge dz$, and recall that ω_0 is a $(p, 0)$ -current on X with values in $E_{\kappa|X}$; cf., Proposition 3.4 and the paragraph preceding it. By dualizing and tensorizing by Ω^N we get the complex

$$0 \leftarrow \mathcal{O}(E_M^*) \otimes \Omega^N \xleftarrow{f_M^* \otimes \text{Id}} \dots \xleftarrow{f_1^* \otimes \text{Id}} \mathcal{O}(E_0^*) \otimes \Omega^N \leftarrow 0$$

with associated cohomology sheaves

$$\mathcal{H}^\ell(\mathcal{O}(E_\bullet^*) \otimes \Omega^N) \simeq \mathcal{E}\text{xt}_\mathcal{O}^\ell(\Omega_X^{n-p}, \Omega^N).$$

Let $\xi \in \mathcal{O}(E_\kappa^*)$ be such that $f_{\kappa+1}^*\xi = 0$. Then

$$\bar{\partial}(\xi \cdot i_*\omega_0) = \xi \cdot \bar{\partial}R_\kappa \wedge dz = \xi \cdot f_{\kappa+1}R_{\kappa+1} \wedge dz = f_{\kappa+1}^*\xi \cdot R_{\kappa+1} \wedge dz = 0,$$

and it follows that the current $i^*\xi \cdot \omega_0$ is $\bar{\partial}$ -closed on X . Hence, $i^*\xi \cdot \omega_0$ is a section of ω_X^p . If $\xi = f_\kappa^*\xi'$ one checks in a similar way that $i^*\xi \cdot \omega_0 = 0$ and we see that we have a mapping

$$(4.1) \quad \begin{aligned} \mathcal{H}^\kappa(\mathcal{O}(E_\bullet^*) \otimes \Omega^N) &\longrightarrow \omega_X^p \\ [\xi] \otimes dz &\mapsto i^*\xi \cdot \omega_0. \end{aligned}$$

Proposition 4.1. *The mapping (4.1) is an isomorphism and it induces a natural isomorphism*

$$\mathcal{E}\text{xt}_\mathcal{O}^\kappa(\Omega_X^{n-p}, \Omega^N) \simeq \omega_X^p.$$

Proof. Let φ be a section of ω_X^p . Then $i_*\varphi$ is a $\bar{\partial}$ -closed $(\kappa + p, \kappa)$ -current in \mathbb{B} and it induces a map $\Omega^{n-p} \rightarrow \mathcal{CH}_X^N$ by

$$(4.2) \quad \psi \mapsto i_*\varphi \wedge \psi,$$

whose kernel clearly contains \mathcal{J}_X^{n-p} . Hence, (4.2) induces a map $\Omega^{n-p}/\mathcal{J}_X^{n-p} \rightarrow \mathcal{CH}_X^N$. Thus, we get a map $\omega_X^p \rightarrow \mathcal{H}\text{om}_\mathcal{O}(\Omega_X^{n-p}, \mathcal{CH}_X^N)$, which one easily checks is injective. In view of (4.1) we get a commutative diagram

$$(4.3) \quad \begin{array}{ccc} \mathcal{H}^\kappa(\mathcal{O}(E_\bullet^*) \otimes \Omega^N) & \xrightarrow{\quad} & \omega_X^p \\ & \searrow & \downarrow \\ & & \mathcal{H}\text{om}_\mathcal{O}(\Omega_X^{n-p}, \mathcal{CH}_X^N), \end{array}$$

where the diagonal map is the composition, i.e., the map given by $[\xi] \otimes dz \mapsto \xi \cdot R_\kappa \wedge dz$, where we here view $R_\kappa \wedge dz$ as a $\text{Hom}(E_0, E_\kappa)$ -valued (N, κ) -current. By [6, Theorem 1.5] this map is an isomorphism and since the vertical map is injective it follows that both the horizontal map and the vertical map are isomorphisms. From *ibid.* we also know that the diagonal map is independent of the choices of Hermitian resolution of Ω_X^{n-p} and of dz . \square

D. Barlet has recently found an elegant algebraic proof of the isomorphism $\mathcal{E}\text{xt}_\mathcal{O}^\kappa(\Omega_X^{n-p}, \Omega^N) \simeq \omega_X^p$ of Proposition 4.1 that he has communicated to us and generously let us include here.

Alternative proof of Proposition 4.1. In this proof we construe Ω_X^{n-p} as the sheaf of germs of Kähler-Grothendieck differential $n - p$ -forms; $\mathcal{E}\text{xt}_\mathcal{O}^\kappa(\Omega_X^{n-p}, \Omega^N)$ is not affected by this (but, in general, higher $\mathcal{E}\text{xt}$ -sheaves are). Let $\mathcal{G} := (d\mathcal{J}_X^0 \wedge \Omega^{n-p-1}) \cap$

$(\mathcal{J}_X^0 \Omega^{n-p})$, let $\mathcal{F} := d\mathcal{J}_X^0 \wedge \Omega^{n-p-1}/\mathcal{G}$, and notice that \mathcal{F} and \mathcal{G} are \mathcal{O}_X -modules. We have a natural short exact sequence of \mathcal{O}_X -modules in \mathbb{B}

$$0 \rightarrow \mathcal{F} \longrightarrow \mathcal{O}_X \otimes \Omega^{n-p} \longrightarrow \Omega_X^{n-p} \rightarrow 0,$$

and applying the functor $\mathcal{H}\text{om}_{\mathcal{O}}(-, \Omega^N)$ one obtains a long exact sequence of derived functors, i.e., of $\mathcal{E}\text{xt}$ -sheaves. Since $\text{codim } X = \kappa$ these sheaves vanish until level κ and in particular one gets the exact sequence

$$0 \rightarrow \mathcal{E}\text{xt}_{\mathcal{O}}^{\kappa}(\Omega_X^{n-p}, \Omega^N) \longrightarrow \mathcal{E}\text{xt}_{\mathcal{O}}^{\kappa}(\mathcal{O}_X \otimes \Omega^{n-p}, \Omega^N) \xrightarrow{b} \mathcal{E}\text{xt}_{\mathcal{O}}^{\kappa}(\mathcal{F}, \Omega^N).$$

Since Ω^{n-p} is a free \mathcal{O} -module and since $\mathcal{E}\text{xt}_{\mathcal{O}}^{\kappa}(\mathcal{O}_X, \Omega^N) \simeq i_* \omega_X^n$ by [13, Lemma 4], one has

$$\mathcal{E}\text{xt}_{\mathcal{O}}^{\kappa}(\mathcal{O}_X \otimes \Omega^{n-p}, \Omega^N) \simeq \mathcal{H}\text{om}_{\mathcal{O}}(\Omega^{n-p}, \mathcal{E}\text{xt}_{\mathcal{O}}^{\kappa}(\mathcal{O}_X, \Omega^N)) \simeq \mathcal{H}\text{om}_{\mathcal{O}}(\Omega^{n-p}, i_* \omega_X^n).$$

Since $\omega_X^p \simeq \mathcal{H}\text{om}_{\mathcal{O}_X}(\Omega_X^{n-p}, \omega_X^n)$ by [13, Proposition 3], we will be done if we can show that the kernel of the map b above consists of those homomorphisms $\Omega^{n-p} \rightarrow i_* \omega_X^n$ which in fact are homomorphisms $\Omega_X^{n-p} \rightarrow \omega_X^n$; since $\mathcal{J}_X^0 i_* \omega_X^n = 0$, a homomorphism $\Omega^{n-p} \rightarrow i_* \omega_X^n$ is a homomorphism $\Omega_X^{n-p} \rightarrow \omega_X^n$ if and only if it vanishes on $d\mathcal{J}_X^0 \wedge \Omega^{n-p-1}$. To understand the map b one can for instance use that $(\mathcal{C}^{N,\bullet}, \bar{\partial})$, where $\mathcal{C}^{N,\bullet}$ is the sheaf of germs of (N, \bullet) -currents in \mathbb{B} , is a resolution of Ω^N by stalk-wise injective sheaves. In fact, then

$$\mathcal{E}\text{xt}_{\mathcal{O}}^{\kappa}(\mathcal{O}_X \otimes \Omega^{n-p}, \Omega^N) \simeq \mathcal{H}^{\kappa}(\mathcal{H}\text{om}_{\mathcal{O}}(\Omega^{n-p}, \mathcal{H}\text{om}_{\mathcal{O}}(\mathcal{O}_X, \mathcal{C}^{N,\bullet})), \bar{\partial})$$

and, since $\mathcal{F} = \mathcal{O}_X \otimes \mathcal{F}$,

$$\mathcal{E}\text{xt}_{\mathcal{O}}^{\kappa}(\mathcal{F}, \Omega^N) \simeq \mathcal{H}^{\kappa}(\mathcal{H}\text{om}_{\mathcal{O}}(\mathcal{F}, \mathcal{H}\text{om}_{\mathcal{O}}(\mathcal{O}_X, \mathcal{C}^{N,\bullet})), \bar{\partial})$$

and the map b is induced by restricting homomorphisms defined on Ω^{n-p} to the subsheaf $d\mathcal{J}_X^0 \wedge \Omega^{n-p-1}$. \square

Notice that it follows from Proposition 4.1 that ω_X^p is coherent, which, as mentioned above, also is proved in [13]. That the vertical map in (4.3) is an isomorphism can be seen as a version of/complement to [13, Lemma 4]. In fact, in view of [6, Theorem 1.5], in our terminology that lemma says that ω_X^p , via i_* , is isomorphic to the sheaf of germs of Coleff-Herrera currents μ in \mathbb{B} of bidegree $(\kappa + p, \kappa)$ such that $\mathcal{J}_X^0 \mu = 0$ and $d\mathcal{J}_X^0 \wedge \mu = 0$, i.e., such that $\mu = i_* \tau$ for some $(p, 0)$ -current τ on X ; cf. the paragraph after the dimension principle in Section 2.2. On the other hand, that the vertical map in (4.3) is an isomorphism means that ω_X^p , via i_* , is isomorphic to the sheaf of germs of Coleff-Herrera currents μ in \mathbb{B} of bidegree $(\kappa + p, \kappa)$ such that $\mathcal{J}_X^{n-p} \wedge \mu = 0$.

That the vertical map in (4.3) is an isomorphism also implies that the map

$$(4.4) \quad \omega_X^p \rightarrow \mathcal{H}\text{om}_{\mathcal{O}_X}(\Omega_X^{n-p}, \omega_X^n), \quad \mu \mapsto (\varphi \mapsto \mu \wedge \varphi)$$

is an isomorphism, which is [13, Proposition 3]; one may construe Ω_X^{n-p} in (4.4) also as the Kähler-Grothendieck differential $n - p$ -forms. It is clear that (4.4) is injective, and if λ is a homomorphism $\Omega_X^{n-p} \rightarrow \omega_X^n$, then $i_* \circ \lambda$ is a homomorphism $\Omega_X^{n-p} \rightarrow \mathcal{C}\mathcal{H}_X^N$. Since the vertical map in (4.3) is an isomorphism there is a $\mu \in \omega_X^p$ such that $i_* \circ \lambda(\varphi) = i_*(\mu \wedge \varphi)$ and thus (4.4) is surjective.

We summarize some results in

Proposition 4.2. *Let X be a reduced complex space of pure dimension n .*

- (i) A meromorphic p -form μ on X is in ω_X^p if and only if $\bar{\partial}(\mu \wedge \varphi) = 0$ for all $\varphi \in \Omega_X^{n-p}$.
- (ii) The pairing $\omega_X^p \times \Omega_X^{n-p} \rightarrow \omega_X^n$, $(\mu, \varphi) \mapsto \mu \wedge \varphi$ is non-degenerate.
- (iii) If Ω_X^{n-p} is Cohen-Macaulay, then ω_X^p is Cohen-Macaulay and a meromorphic $n-p$ -form φ on X is in Ω_X^{n-p} if and only if $\bar{\partial}(\mu \wedge \varphi) = 0$ for all $\mu \in \omega_X^p$.

Proof. Part (i) is immediate since (4.4) is an isomorphism. Part (ii) clearly holds on X_{reg} . Since Ω_X^{n-p} has pure dimension and currents in ω_X^p have the SEP the non-degeneracy extends to X . To see part (iii) we recall first the well-known fact that if Ω_X^{n-p} is Cohen-Macaulay (and $\mathcal{O}(E_\bullet)$ is a minimal resolution of Ω_X^{n-p}), then $\mathcal{H}^\kappa(\mathcal{O}(E_\bullet^*) \otimes \Omega^N)$ is Cohen-Macaulay. The first claim of part (iii) thus follows from Proposition 4.1. For the second claim, recall that $\omega = \omega_0$ is $\bar{\partial}$ -closed since Ω_X^{n-p} is Cohen-Macaulay. Thus, by Theorem 5.4 below, φ is in Ω_X^{n-p} if and only if $\bar{\partial}(\varphi \wedge \omega_0) = 0$. Now, ω_0 is an $E_{\kappa|X}$ -valued section of ω_X^p , that is, a tuple of sections of ω_X^p , and we are done. \square

Remark 4.3. Part (ii) is a concrete manifestation of [6, Theorem 1.2] for the sheaf $\mathcal{F} = \Omega_X^{n-p}$.

We conclude this section by noticing that it follows from [13] that, on a normal complex space X , ω_X^p coincides with the reflexive hull of the p^{th} exterior power of the sheaf of Kähler differentials, denoted $\Omega_X^{[p]}$ as in [25]. In fact, if X is any pure-dimensional reduced complex space and $A \subset X$ is an analytic subset, then, by [13, p. 195], $\mathcal{H}_A^0(\omega_X^p) = 0$ if $\text{codim}_X A \geq 1$ (i.e., sections of ω_X^p have the SEP) and $\mathcal{H}_A^1(\omega_X^p) = 0$ if $\text{codim}_X A \geq 2$. It follows that the restriction map $\omega^p(\mathcal{U}) \rightarrow \omega^p(\mathcal{U} \setminus A)$ is bijective for any open $\mathcal{U} \subset X$ and analytic $A \subset \mathcal{U}$ with $\text{codim}_{\mathcal{U}} A \geq 2$. On a normal complex space, a coherent sheaf is reflexive if and only if it has this extension property and is torsion free, see [27, Proposition 1.6]. Since ω_X^p and $\Omega_X^{[p]}$ coincide on X_{reg} and the singular locus has codimension ≥ 2 on a normal complex space it follows that $\omega_X^p \simeq \Omega_X^{[p]}$ if X is normal.

5. INTEGRAL OPERATORS ON AN ANALYTIC SUBSET

Let $D \subset \mathbb{C}^N$ be a domain (not necessarily pseudoconvex at this point), let $k(\zeta, z)$ be an integrable $(N, N-1)$ -form in $D \times D$, and let $p(\zeta, z)$ be a smooth (N, N) -form in $D \times D$. Assume that k and p satisfy the equation of currents

$$(5.1) \quad \bar{\partial}k(\zeta, z) = [\Delta^D] - p(\zeta, z)$$

in $D \times D$, where $[\Delta^D]$ is the current of integration along the diagonal. Applying this current equation to test forms $\psi(z) \wedge \varphi(\zeta)$ it is straightforward to verify that for any compactly supported (p, q) -form φ in D one has the following Koppelman formula

$$\varphi(z) = \bar{\partial}_z \int_{D_\zeta} k(\zeta, z) \wedge \varphi(\zeta) + \int_{D_\zeta} k(\zeta, z) \wedge \bar{\partial}\varphi(\zeta) + \int_{D_\zeta} p(\zeta, z) \wedge \varphi(\zeta).$$

In [1] Andersson introduced a very flexible method of producing solutions to (5.1). Let $\eta = (\eta_1, \dots, \eta_N)$ be a holomorphic tuple in $D \times D$ that defines the diagonal and let Λ_η be the exterior algebra spanned by $\Lambda^{0,1}T^*(D \times D)$ and the $(1, 0)$ -forms $d\eta_1, \dots, d\eta_N$. On forms with values in Λ_η interior multiplication with $2\pi i \sum \eta_j \partial/\partial \eta_j$, denoted δ_η , is defined; put $\nabla_\eta = \delta_\eta - \bar{\partial}$.

Let s be a smooth $(1,0)$ -form in Λ_η such that $|s| \lesssim |\eta|$ and $|\eta|^2 \lesssim |\delta_\eta s|$ and let $B = \sum_{k=1}^N s \wedge (\bar{\partial}s)^{k-1}/(\delta_\eta s)^k$. It is proved in [1] that then $\nabla_\eta B = 1 - [\Delta^D]$. Identifying terms of top degree we see that $\bar{\partial}B_{N,N-1} = [\Delta^D]$ and we have found a solution to (5.1). For instance, if we take $s = \partial|\zeta - z|^2$ and $\eta = \zeta - z$, then the resulting B is sometimes called the full Bochner-Martinelli form and the term of top degree is the classical Bochner-Martinelli kernel.

A smooth section $g(\zeta, z) = g_{0,0} + \cdots + g_{N,N}$ of Λ_η , defined for $z \in D' \subset D$ and $\zeta \in D$, such that $\nabla_\eta g = 0$ and $g_{0,0} \restriction_{\Delta^D} = 1$ is called a *weight* with respect to $z \in D'$. It follows that $\nabla_\eta(g \wedge B) = g - [\Delta^D]$ and, identifying terms of bidegree $(N, N-1)$, we get that

$$(5.2) \quad \bar{\partial}(g \wedge B)_{N,N-1} = [\Delta^D] - g_{N,N}$$

in $D_\zeta \times D'_z$ and hence another solution to (5.1). If D is pseudoconvex and K is a holomorphically convex compact subset, then one can find a weight g with respect to z in some neighborhood $D' \Subset D$ of K such that $z \mapsto g(\zeta, z)$ is holomorphic in D' and $\zeta \mapsto g(\zeta, z)$ has compact support in D ; see, e.g., [3, Example 2] or [8, Example 5.1] in case $D = \mathbb{B}$. We will also have use for weights with values in a certain type of vector bundle, cf. [22] and [3]. Let $V \rightarrow D$ be a vector bundle, let $\pi_\zeta: D_\zeta \times D_z \rightarrow D_\zeta$ and $\pi_z: D_\zeta \times D_z \rightarrow D_z$ be the natural projections and set $V_z \otimes V_\zeta^* := \pi_z^* V \otimes \pi_\zeta^* V^*$. Then a weight may take values in $V_z \otimes V_\zeta^* \simeq \text{Hom}(V_\zeta, V_z)$; it should satisfy the same properties but with the condition $g_{0,0} \restriction_{\Delta^D} = 1$ replaced by $g_{0,0} \restriction_{\Delta^D} = \text{Id}_V$. If g is a weight with values in $V_z \otimes V_\zeta^*$ then (5.2) holds with $[\Delta^D]$ replaced by $\text{Id}_V \otimes [\Delta^D]$.

Let \tilde{X} be an analytic subset of pure codimension κ of a neighborhood of \overline{D} , where D now is assumed to be strictly pseudoconvex, and set $X = \tilde{X} \cap D$. Let (2.6) be a free resolution of Ω_X^p in D and let $U = U(\zeta)$ and $R = R(\zeta)$ be the associated currents (for some choice of Hermitian metrics on the E_k 's). Let $E_k^z := \pi_z^* E_k$ and similarly for E_k^ζ . One can find Hefer morphisms H_k^ℓ , which are $\text{Hom}(E_k^\zeta, E_\ell^z)$ -valued $(k-\ell, 0)$ -forms depending holomorphically on $(\zeta, z) \in D \times D$ such that

$$H_k^k \restriction_{\Delta_D} = \text{Id}_{E_k} \quad \text{and} \quad \delta_\eta H_k^\ell = H_{k-1}^\ell f_k - f_{\ell+1}(z) H_k^{\ell+1}, \quad k > \ell,$$

where $f_k = f_k(\zeta)$; see [3, Proposition 5.3]. Let $F = F(\zeta)$ be a holomorphic tuple such that $X = \{F = 0\}$ and set $\chi^\epsilon := \chi(|F|^2/\epsilon)$; we regularize U and R as in Section 2 so that $U^\epsilon := \chi^\epsilon u$ and

$$R^\epsilon := \text{Id}_E - \nabla U^\epsilon = (1 - \chi^\epsilon) \text{Id}_E + \bar{\partial} \chi^\epsilon \wedge u.$$

We write U_k^ϵ and R_k^ϵ for the parts of U^ϵ and R^ϵ that take values in $\text{Hom}(E_0, E_k)$ and we define

$$G^\epsilon := \sum_{k \geq 0} H_k^0 R_k^\epsilon + f_1(z) \sum_{k \geq 1} H_k^1 U_k^\epsilon,$$

which one can check is a weight with values in $\text{Hom}(E_0^\zeta, E_0^z)$.

Letting g be any scalar-valued weight with respect to, say, $z \in D' \subset D$ it follows that $G^\epsilon \wedge g$ is a $\text{Hom}(E_0^\zeta, E_0^z)$ -valued weight and (5.2) holds with g replaced by $G^\epsilon \wedge g$ and $[\Delta^D]$ replaced by $\text{Id}_{E_0} \otimes [\Delta^D]$. Let $\nabla^z = f(z) - \bar{\partial}$ and let ∇_{End}^z be the corresponding endomorphism-valued operator. Then, recalling that $\nabla_{\text{End}}^z R(z) = 0$ and noticing that $\nabla_{\text{End}}^z(G^\epsilon \wedge g \wedge B) = -\bar{\partial}(G^\epsilon \wedge g \wedge B)$ since $f(z) \restriction_{E_0} = 0$, we get

$$(5.3) \quad \begin{aligned} & -\nabla_{\text{End}}^z (R(z) \wedge dz \wedge (G^\epsilon \wedge g \wedge B)_{N,N-1}) \\ & = R(z) \wedge dz \wedge [\Delta^D] - R(z) \wedge dz \wedge (G^\epsilon \wedge g)_{N,N}. \end{aligned}$$

Notice that $R(z) \wedge [\Delta^D]$ and $R(z) \wedge B$ are well-defined; they are simply tensor products of currents since z and $\zeta - z$ are independent variables on $D \times D$. Since $R(z)f_1(z) = 0$, (5.3) becomes

$$(5.4) \quad \begin{aligned} & -\nabla_{\text{End}}^z (R(z) \wedge dz \wedge (HR^\epsilon \wedge g \wedge B)_{N,N-1}) \\ &= R(z) \wedge dz \wedge [\Delta^D] - R(z) \wedge dz \wedge (HR^\epsilon \wedge g)_{N,N}, \end{aligned}$$

where $HR^\epsilon := \sum_{k \geq 0} H_k^0 R_k^\epsilon$. Let $\iota: X \simeq \Delta^X \hookrightarrow X \times X$ be the diagonal embedding and let $\mathbf{i}: X \times X \hookrightarrow D \times D$ be the inclusion. By Proposition 3.4 we have

$$(5.5) \quad \mathbf{i}_* \iota_* \omega = R(z) \wedge dz \wedge [\Delta^D],$$

where ω is the $n-p$ -structure form corresponding to R .

Consider now the term $(HR^\epsilon \wedge g)_{N,N}$. Noticing that R^ϵ contains no $d\eta_j$ we see that

$$(5.6) \quad (HR^\epsilon \wedge g)_{N,N} = \tilde{p}(\zeta, z) \wedge R^\epsilon \wedge d\eta,$$

for some $\text{Hom}(E^\zeta, E_0^z)$ -valued form $\tilde{p}(\zeta, z)$ that is smooth for $(\zeta, z) \in D \times D'$; if g is chosen holomorphic in z (respectively ζ), then \tilde{p} is holomorphic in z (respectively ζ). To further reveal the structure of \tilde{p} , let $\varepsilon_1, \dots, \varepsilon_N$ be a frame for an auxiliary trivial vector bundle $F \rightarrow D \times D$, replace each occurrence of $d\eta_j$ in H and g by ε_j , and denote the result by \hat{H} and \hat{g} . We get

$$(5.7) \quad \tilde{p}(\zeta, z) \wedge R^\epsilon \wedge \varepsilon = (\hat{H} R^\epsilon \wedge \hat{g})_{N,N} = \sum_{k \geq 0} \hat{H}_k^0 R_k^\epsilon \wedge \hat{g}_{N-k, N-k} = \sum_{k \geq 0} \tilde{p}_k(\zeta, z) \wedge R_k^\epsilon \wedge \varepsilon,$$

where $\tilde{p}_k(\zeta, z) = \pm \epsilon^* \lrcorner \hat{H}_k^0 \wedge \hat{g}_{N-k, N-k}$ is a smooth $(0, N-k)$ -form in $D \times D'$ with values in $\text{Hom}(E_k^\zeta, E_0^z)$; it is holomorphic in z (or ζ) if g is chosen so. For degree reasons it follows that

$$(5.8) \quad R(z) \wedge dz \wedge (HR^\epsilon \wedge g)_{N,N} = R(z) \wedge dz \wedge \sum_{k \geq 0} \tilde{p}_k(\zeta, z) \wedge R_k^\epsilon \wedge d\zeta.$$

Since $R(z) \wedge R$ is well-defined (as a tensor product) we may set $\epsilon = 0$ in (5.8) and since $R = R_\kappa + R_{\kappa+1} + \dots$ we then sum only over $k \geq \kappa$. In view of Proposition 3.4 it follows that

$$(5.9) \quad \lim_{\epsilon \rightarrow 0} R(z) \wedge dz \wedge (HR^\epsilon \wedge g)_{N,N} = \mathbf{i}_* \omega(z) \wedge p(\zeta, z),$$

where

$$p(\zeta, z) := \sum_{k \geq \kappa} \mathbf{i}^* \tilde{p}_k(\zeta, z) \wedge \omega_{k-\kappa}(\zeta) = \sum_{k \geq \kappa} \pm \mathbf{i}^* (\varepsilon^* \lrcorner \hat{H}_k^0 \wedge \hat{g}_{N-k, N-k}) \wedge \omega_{k-\kappa}(\zeta).$$

We here, and in the following, view \tilde{p}_k not as $(0, N-k)$ -form with values in $\text{Hom}(E_k^\zeta, E_0^z)$ but as a $(p, N-k)$ -form with values in $(E_k^\zeta)^*$; cf. the paragraph preceding Proposition 3.4. Thus, $p(\zeta, z)$ is a scalar valued almost semi-meromorphic current on $X \times X'$ of bidegree (n, n) such that $z \mapsto p(\zeta, z)$ is, or rather, has a natural extension that is smooth in D (or holomorphic if $z \mapsto g(\zeta, z)$ is); notice that $p(\zeta, z)$ has degree p in dz_j and degree $n-p$ in $d\zeta_j$.

We proceed in an analogous way with the current $R(z) \wedge dz \wedge (HR^\epsilon \wedge g \wedge B)_{N,N-1}$ and we get, cf. (5.8), that

$$(5.10) \quad R(z) \wedge dz \wedge (HR^\epsilon \wedge g \wedge B)_{N,N-1} = R(z) \wedge dz \wedge \sum_{j \geq 0} \tilde{k}_j(\zeta, z) \wedge R_j^\epsilon \wedge d\zeta,$$

where $\tilde{k}_j(\zeta, z) := \pm \varepsilon^* \lrcorner \hat{H}_j^0 \wedge (\hat{g} \wedge \hat{B})_{N-j, N-j-1}$ is a $(0, N-j-1)$ -form with values in $\text{Hom}(E_j^\zeta, E_0^z)$. From Section 2 we know that the limit as $\epsilon \rightarrow 0$ of (5.10) exists and yields a pseudomeromorphic current in $D \times D'$. Moreover, precisely as in [8, Lemma 5.2] one shows that

$$\lim_{\epsilon \rightarrow 0} R(z) \wedge dz \wedge (HR^\epsilon \wedge g \wedge B)_{N, N-1} = \lim_{\epsilon \rightarrow 0} R(z) \wedge dz \wedge (HR \wedge g \wedge B^\epsilon)_{N, N-1},$$

where $B^\epsilon := \chi(|\eta|^2/\epsilon)B$, holds in the sense of current on $(D \setminus X_{sing}) \times (D' \setminus X_{sing})$. In view of (5.10) and Proposition 3.4 we thus get

$$(5.11) \quad \lim_{\epsilon \rightarrow 0} R(z) \wedge dz \wedge (HR^\epsilon \wedge g \wedge B)_{N, N-1} = \lim_{\epsilon \rightarrow 0} \chi(|\eta|^2/\epsilon) i_* \omega(z) \wedge k(\zeta, z)$$

in $(D \setminus X_{sing}) \times (D' \setminus X_{sing})$, where

$$(5.12) \quad k(\zeta, z) := \sum_{j \geq \kappa} i^* \tilde{k}_j(\zeta, z) \wedge \omega_{j-\kappa}(\zeta) = \pm \sum_{j \geq \kappa} i^* (\varepsilon^* \lrcorner \hat{H}_j^0 \wedge (\hat{g} \wedge \hat{B})_{N-j, N-j-1}) \wedge \omega_{j-\kappa}(\zeta).$$

As with $\tilde{p}_j(\zeta, z)$, we here and in the following view $\tilde{k}_j(\zeta, z)$ as a $(p, N-j-1)$ -form with values in $(E_j^\zeta)^*$ so that $k(\zeta, z)$ becomes a scalar valued almost semi-meromorphic $(n, n-1)$ -current on $X \times X'$; the degree in dz_j being p and the degree in $d\zeta_j$ being $n-p$. Recall that $B_{\ell, \ell-1} = s \wedge (\bar{\partial}s)^{\ell-1}/(\delta_\eta s)^\ell$ and that $|s| \lesssim |\eta|$ and $|\eta|^2 \lesssim |\delta_\eta s|$. Since $\hat{B}_{\ell, \ell-1}$, $\ell = 1, \dots, n$ are the only components of \hat{B} that enters in the expression for $k(\zeta, z)$ it follows that $k(\zeta, z)$ is integrable on $X_{reg} \times X'_{reg}$. Hence, the limit on the right hand side of (5.11) is just the locally integrable form $k(\zeta, z) \wedge \omega(\zeta)$ on $X_{reg} \times X'_{reg}$. From (5.4), (5.5), (5.9), and (5.11) we thus see that

$$(5.13) \quad -\nabla \omega(z) \wedge k(\zeta, z) = \iota_* \omega - \omega(z) \wedge p(\zeta, z)$$

as currents on $X_{reg} \times X'_{reg}$, where ∇ here means the endomorphism-version of $f(z) \restriction_X -\bar{\partial}$. Since R is ∇_{End} -closed it follows that $\omega(z)$ is ∇ -closed and so the left hand side of (5.13) equals $\omega(z) \wedge \bar{\partial}k(\zeta, z)$. By Lemma 3.5 we have thus proved

Proposition 5.1. *In $X_{reg} \times X'_{reg}$ we have that $\bar{\partial}k(\zeta, z) = [\Delta^X] - p(\zeta, z)$ as currents.*

The following technical lemma corresponds to [8, Lemma 6.4]; cf. also [41, Proposition 4.3 (ii)].

Lemma 5.2. *Let ω be any $n-p$ -structure form and let k_j , $j = 1, \dots, \nu$, be given by (5.12) for possibly different choices of H 's, g 's, B 's, and $n-p$ -structure forms ω 's. Then*

$$(5.14) \quad T := \omega(z^\nu) \wedge k_\nu(z^{\nu-1}, z^\nu) \wedge k_{\nu-1}(z^{\nu-2}, z^{\nu-1}) \wedge \dots \wedge k_1(z^0, z^1)$$

is an almost semi-meromorphic current on $X^{\nu+1}$ and, if $h = h(z^j)$ is a generically non-vanishing holomorphic tuple on X_{z^j} then $\bar{\partial}\chi(|h|^2/\epsilon) \wedge T \rightarrow 0$ as $\epsilon \rightarrow 0$.

Proof. In view of Section 2.2, T is almost semi-meromorphic since each factor is. For the second statement we proceed by induction over ν . If $\nu = 0$ then $T = \omega$. Since R is ∇_{End} -closed it follows that $\bar{\partial}\omega = f \restriction_X \omega$, which has the SEP. From Lemma 2.2 we see that $\bar{\partial}\chi(|h|^2/\epsilon) \wedge \omega \rightarrow 0$ as $\epsilon \rightarrow 0$.

Now assume that $\nu \geq 1$ and assume that the lemma holds for $\nu \leq \ell-1$. Let T be given by (5.14) with $\nu = \ell$. If $z^k \neq z^{k-1}$ then k_k is a smooth form times some $n-p$ -structure form $\tilde{\omega}(z^{k-1})$. It follows that $T = T' \wedge T''$ is the tensor product of two currents T' and T'' of the form (5.14) with $\nu < \ell$ as long as $z^k \neq z^{k-1}$. By the

induction hypothesis $\bar{\partial}\chi(|h|^2/\epsilon) \wedge T \rightarrow 0$ where $z^k \neq z^{k-1}$. Thus we see that the support of the pseudomeromorphic current $\lim_{\epsilon \rightarrow 0} \bar{\partial}\chi(|h|^2/\epsilon) \wedge T$ must be contained in $\{z^0 = \dots = z^\nu\} \cap \{h = 0\}$, which has codimension $\geq n\nu + 1$. Let T_r be the term of T corresponding to the term ω_r of ω . Then the pseudomeromorphic current $\lim_{\epsilon \rightarrow 0} \bar{\partial}\chi(|h|^2/\epsilon) \wedge T_0$ has bidegree $(n\nu + n - p, n\nu + 1 - \nu)$ and thus must vanish by the dimension principle. By Proposition 3.4 (ii) there is an almost semi-meromorphic $(0, 1)$ -current α_1 that is smooth outside of $Z_{\kappa+1}$ and such that $\omega_1 = \alpha_1 \omega_0$. From what we have just showed it follows that the pseudomeromorphic $(n\nu + n - p, n\nu + 2 - \nu)$ -current $\lim_{\epsilon \rightarrow 0} \bar{\partial}\chi(|h|^2/\epsilon) \wedge T_1$ has support in $\{z^0 = \dots = z^\nu\} \cap \{h = 0\} \cap \{z^j \in Z_{\kappa+1}\}$, which has codimension $\geq n\nu + 2$. Hence that current vanishes by the dimension principle. Continuing in this way the lemma follows. \square

5.1. The integral operators \mathcal{K} and \mathcal{P} on $(p, *)$ -forms. In order to construct the integral operators \mathcal{K} we choose the weight g in the definitions of $p(\zeta, z)$ and $k(\zeta, z)$ to be a weight with respect to $z \in D' \Subset D$ such that $\zeta \mapsto g(\zeta, z)$ has compact support in D . Let φ be a pseudomeromorphic (p, q) -current on X . In view of Section 2.2, $k(\zeta, z) \wedge \varphi(\zeta)$ and $p(\zeta, z) \wedge \varphi(\zeta)$ are well-defined pseudomeromorphic currents in $X_\zeta \times X'_z$, where $X' = X \cap D'$. Let $\pi^z: X_\zeta \times X_z \rightarrow X_z$ be the natural projection and set

$$(5.15) \quad \mathcal{K}\varphi(z) := \pi_*^z k(\zeta, z) \wedge \varphi(\zeta), \quad \mathcal{P}\varphi(z) := \pi_*^z p(\zeta, z) \wedge \varphi(\zeta).$$

Since $\zeta \mapsto g(\zeta, z)$ has compact support in D it follows that $\mathcal{K}\varphi$ and $\mathcal{P}\varphi$ are well-defined pseudomeromorphic currents in X' . Notice that $\mathcal{P}\varphi$ has a natural smooth extension to D' since $z \mapsto p(\zeta, z)$ has; notice also that if φ has the SEP then $\mathcal{K}\varphi$ has the SEP in view of Section 2.2. Moreover, as in [8, Lemma 6.1] one shows that if $\varphi = 0$ in a neighborhood of a point $x \in X'$, or if φ is smooth in a neighborhood of x and $x \in X'_{reg}$, then $\mathcal{K}\varphi$ is smooth in a neighborhood of x .

If φ is a pseudomeromorphic (p, q) -current with compact support in X , then one can choose any weight g in the definitions of $k(\zeta, z)$ and $p(\zeta, z)$ and define $\mathcal{K}\varphi$ and $\mathcal{P}\varphi$ by (5.15); the outcome has the same general properties.

The following proposition is proved in the same way as [8, Proposition 6.3].

Proposition 5.3. *Let $\varphi \in \mathcal{W}^{p,q}(X)$, let ω be the $n-p$ -structure form that enters in the definitions of $k(\zeta, z)$ and $p(\zeta, z)$, and assume that $\bar{\partial}(\omega \wedge \varphi)$ has the SEP. Let g be a weight with respect to $z \in D' \subset D$. If either g has compact support in D_ζ or φ has compact support in X then*

$$\varphi = \bar{\partial}\mathcal{K}\varphi + \mathcal{K}(\bar{\partial}\varphi) + \mathcal{P}\varphi$$

as currents on X'_{reg} .

Notice that the condition that $\bar{\partial}(\omega \wedge \varphi)$ has the SEP implies that $\bar{\partial}\varphi$ has the SEP. In fact, from Section 2.2 we know that $\omega \wedge \varphi$ has the SEP and so, in view of Lemma 2.2, $\bar{\partial}(\omega \wedge \varphi)$ has the SEP if and only if $\bar{\partial}\chi(|h|^2/\epsilon) \wedge \omega \wedge \varphi \rightarrow 0$ for all generically non-vanishing h . In particular, $\bar{\partial}\chi(|h|^2/\epsilon) \wedge \omega_0 \wedge \varphi \rightarrow 0$ and so, by Lemma 3.5, $\bar{\partial}\chi(|h|^2/\epsilon) \wedge \varphi \rightarrow 0$. By Lemma 2.2 again we conclude that $\bar{\partial}\varphi$ has the SEP.

From Proposition 5.3 it is easy to prove the following residue criterion for a meromorphic p -form to be strongly holomorphic. Recall the operator $\nabla = \oplus_j f_j - \bar{\partial}$ attached to (2.6).

Theorem 5.4. *Let X be a pure n -dimensional analytic subset of some neighborhood of the closure of a strictly pseudoconvex domain $D \subset \mathbb{C}^N$ and let ω be an $n-p$ -structure form on $X \cap D$ corresponding to a resolution (2.6) of Ω_X^p . Then a meromorphic p -form φ on $X \cap D$ is strongly holomorphic if and only if*

$$(5.16) \quad \nabla(\omega \wedge \varphi) = 0.$$

Moreover, if (5.16) holds, $D' \Subset D$, and \mathcal{P} is an integral operator constructed using ω and a weight $g(\zeta, z)$ such that $z \mapsto g(\zeta, z)$ is holomorphic in D' and $\zeta \mapsto g(\zeta, z)$ has compact support in D , then $\mathcal{P}\varphi$ is a holomorphic extension of $\varphi|_{X \cap D'}$ to D' .

Proof. Notice first that if φ is strongly holomorphic then (5.16) holds since $\nabla\omega = 0$.

For the converse, notice that $\omega \wedge \varphi$ has the SEP so that $\chi(|h|^2/\epsilon)\omega \wedge \varphi \rightarrow \omega \wedge \varphi$ for all generically non-vanishing h . Hence, if (5.16) holds, we get

$$0 = \nabla(\omega \wedge \varphi) = \lim_{\epsilon \rightarrow 0} \nabla(\chi(|h|^2/\epsilon)\omega \wedge \varphi) = -\lim_{\epsilon \rightarrow 0} \bar{\partial}\chi(|h|^2/\epsilon) \wedge \omega \wedge \varphi.$$

for all such h . From Lemma 2.2 it thus follows that $\bar{\partial}(\omega \wedge \varphi)$ has the SEP. From the paragraph after Proposition 5.3 it then follows that $\bar{\partial}\varphi$ has the SEP and since φ is holomorphic generically we see that $\bar{\partial}\varphi = 0$. By Proposition 5.3 we get that $\varphi = \mathcal{P}\varphi$ on $X_{reg} \cap D'$. However, both φ and $\mathcal{P}\varphi$ have the SEP so this holds on $X \cap D'$. \square

Theorem 5.4 gives the following geometric criterion for a meromorphic p -form to be strongly holomorphic.

Proposition 5.5. *Let X be a pure n -dimensional reduced complex space and let φ be a meromorphic p -form on X with pole set $P_\varphi \subset X$. Suppose that (i) $\text{codim}_X P_\varphi \geq 2$, and that (ii) $\text{codim}_X S_{n-k}(\Omega_X^p) \cap P_\varphi \geq k+2$ for $k \geq 1$. Then φ is strongly holomorphic.*

Proof. The statement is local so we may assume that X is an analytic subset of a neighborhood of $\overline{\mathbb{B}} \subset \mathbb{C}^N$. Let $\omega = \omega_0 + \dots$ be an $n-p$ -structure form on $X \cap \mathbb{B}$. By Theorem 5.4 we need to show that $\nabla(\omega \wedge \varphi) = 0$. Since ω and φ are almost semi-meromorphic we have $\pm \omega \wedge \varphi = \varphi \wedge \omega = \lim_{\epsilon \rightarrow 0} \chi(|h|^2/\epsilon) \varphi \wedge \omega$, where h is a generically non-vanishing holomorphic function such that $\{h = 0\} \supset P_\varphi$. Thus, since $\nabla\omega = 0$, we see that $\nabla(\omega \wedge \varphi) = \pm \lim_{\epsilon \rightarrow 0} \bar{\partial}\chi(|h|^2/\epsilon) \wedge \varphi \wedge \omega$ and so we need to show that

$$(5.17) \quad \lim_{\epsilon \rightarrow 0} \bar{\partial}\chi(|h|^2/\epsilon) \wedge \varphi \wedge \omega_\ell = 0$$

for $\ell = 0, 1, 2, \dots$. For $\ell = 0$ the left hand side of (5.17) is a pseudomeromorphic $(n, 1)$ -current on X with support contained in P_φ ; hence it vanishes by the dimension principle and assumption (i).

Recall from Section 2.3 the sets Z_k associated with a resolution (2.6) of Ω_X^p and that $S_{N-k}(\Omega_X^p) = Z_k$. Assumption (ii) is thus equivalent to $\text{codim } Z_k \cap P_\varphi \geq k+2$ for $k \geq N-n+1$. Now, assume that (5.17) holds for $\ell = m$. Since, by Proposition 3.4 (ii), ω_{m+1} is a smooth form times ω_m outside of Z_{m+1} it follows that for $\ell = m+1$ the left hand side of (5.17) is a pseudomeromorphic $(n, m+2)$ -current with support contained in $Z_{m+1} \cap P_\varphi$. Thus, (5.17) holds for $\ell = m+1$ by assumption (ii) and the dimension principle. \square

5.2. The integral operators $\check{\mathcal{K}}$ and $\check{\mathcal{P}}$ on $(n-p, *)$ -forms. A general integral operator $\check{\mathcal{K}}$ is constructed by choosing the weight g in the definitions of $k(\zeta, z)$ and $p(\zeta, z)$ to be a weight with respect to $\zeta \in D' \Subset D$ such that $z \mapsto g(\zeta, z)$ has compact support in D . Let ψ be a pseudomeromorphic $(n-p, q)$ -current on X . In the same

way as above $k(\zeta, z) \wedge \varphi(z)$ and $p(\zeta, z) \wedge \varphi(z)$ are well-defined pseudomeromorphic currents in $X'_\zeta \times X_z$ and we set

$$\check{\mathcal{K}}\psi(\zeta) := \pi_*^\zeta k(\zeta, z) \wedge \varphi(z), \quad \check{\mathcal{P}}\psi(\zeta) := \pi_*^\zeta p(\zeta, z) \wedge \varphi(z),$$

which become pseudomeromorphic currents on X' . Notice that $\check{\mathcal{P}}\psi$ has the SEP if ψ has, and moreover, is of the form $\sum_{\ell \geq 0} A_\ell(\zeta) \wedge \omega_\ell(\zeta)$, where A_ℓ is a smooth form with values in $E_{\kappa+\ell}^*$; if g is chosen so that $\zeta \mapsto g(\zeta, z)$ is holomorphic then the A_ℓ are holomorphic. The current $\check{\mathcal{K}}\psi$ has the SEP if ψ has, and it has the form $\sum_{\ell \geq 0} C_\ell(\zeta) \wedge \omega_\ell(\zeta)$, where the C_ℓ take values in $E_{\kappa+\ell}^*$ and are: i) smooth close to $x \in X'$ if $\psi = 0$ close to x , and ii) smooth close to $x \in X'_{reg}$ if ψ is smooth close to x .

As for \mathcal{K} and \mathcal{P} , if ψ happens to have compact support in X then any weight g may be used to define $\check{\mathcal{K}}\psi$ and $\check{\mathcal{P}}\psi$.

Proposition 5.6. *Let $\psi \in \mathcal{W}^{n-p,q}(X)$, assume that $\bar{\partial}\psi \in \mathcal{W}^{n-p,q+1}(X)$, and let g be a weight with respect to $\zeta \in D' \subset D$. If either g has compact support in D_z or ψ has compact support in X then*

$$\psi = \bar{\partial}\check{\mathcal{K}}\psi + \check{\mathcal{K}}(\bar{\partial}\psi) + \check{\mathcal{P}}\psi$$

as currents on X'_{reg} .

This is proved in the same way as [41, Proposition 3.1].

6. THE SHEAVES $\mathcal{A}_X^{p,q}$ AND $\mathcal{B}_X^{n-p,q}$

6.1. The sheaves $\mathcal{A}_X^{p,\bullet}$. Let X be a reduced complex space of pure dimension n . Following [8, Definition 7.1] we say that a (p, q) -current φ on X on an open subset $U \subset X$ is a section of $\mathcal{A}_X^{p,q}$ over U if for every $x \in U$ the germ φ_x can be written as a finite sum of terms

$$(6.1) \quad \xi_\nu \wedge \mathcal{K}_\nu(\cdots \xi_2 \wedge \mathcal{K}_2(\xi_1 \wedge \mathcal{K}_1(\xi_0)) \cdots),$$

where ξ_0 is a smooth $(p, *)$ -form and the ξ_j , $j \geq 1$, are smooth $(0, *)$ -forms such that ξ_j has support where $z \mapsto k_j(\zeta, z)$ is defined.

Proposition 6.1. *The sheaf $\mathcal{A}_X^{p,q}$ has the following properties:*

- (i) $\mathcal{E}_X^{p,q} \subset \mathcal{A}_X^{p,q} \subset \mathcal{W}_X^{p,q}$ and $\oplus_q \mathcal{A}_X^{p,q}$ is a module over $\oplus_q \mathcal{E}_X^{0,q}$,
- (ii) $\mathcal{A}_{X_{reg}}^{p,q} = \mathcal{E}_{X_{reg}}^{p,q}$,
- (iii) for any operator \mathcal{K} on $(p, *)$ -forms as in Section 5.1 $\mathcal{K}: \mathcal{A}_X^{p,q} \rightarrow \mathcal{A}_X^{p,q-1}$,
- (iv) if φ is a section of $\mathcal{A}_X^{p,q}$ and ω is any $n - p$ -structure form, then $\bar{\partial}(\omega \wedge \varphi)$ has the SEP.

Proof. (i), (ii), and (iii) are immediate from the definition of $\mathcal{A}_X^{p,q}$ and the general properties of the \mathcal{K} -operators in Section 5.1. To prove (iv) we may assume that φ is of the form (6.1). Then $\omega \wedge \varphi$ is a push-forward of $T \wedge \xi$, where T is of the form (5.14) and ξ is a smooth form on $X^{\nu+1}$. Choosing $h = h(z^\nu)$ in Lemma 5.2 it follows that $\bar{\partial}\chi(|h|^2/\epsilon) \wedge \omega \wedge \varphi \rightarrow 0$ as $\epsilon \rightarrow 0$ and so, by Lemma 2.2, $\bar{\partial}(\omega \wedge \varphi)$ has the SEP. \square

Proof of Theorem 1.3. Let $D'' \Subset D$ be a strictly pseudoconvex neighborhood of \overline{D}' and carry out the construction of $k(\zeta, z)$ and $p(\zeta, z)$ in Section 5 in $D'' \times D''$ using a weight $g(\zeta, z)$ with respect to $z \in D'$ such that $z \mapsto g(\zeta, z)$ is holomorphic in D' and $\zeta \mapsto g(\zeta, z)$ has compact support in D'' . Notice that then $\mathcal{P}\varphi$ is holomorphic and that g , and hence also $p(\zeta, z)$, has bidegree $(*, 0)$ in the z -variables so that $\mathcal{P}\varphi = 0$ if

φ has bidegree (p, q) with $q \geq 1$. Let $\varphi \in \mathcal{A}^{p,q}(X)$. By Proposition 6.1 (iv), $\bar{\partial}(\omega \wedge \varphi)$ has the SEP and so Proposition 5.3 shows that

$$(6.2) \quad \varphi = \bar{\partial}\mathcal{K}\varphi + \mathcal{K}(\bar{\partial}\varphi) + \mathcal{P}\varphi$$

in the sense of currents on X'_{reg} . Now, $\mathcal{K}\varphi \in \mathcal{A}^{p,q-1}(X')$ by Proposition 6.1 (iii). Hence, by Proposition 6.1 (iv) and the comment after Proposition 5.3, $\bar{\partial}\mathcal{K}\varphi$ has the SEP. In the same way $\bar{\partial}\varphi$ has the SEP and so $\mathcal{K}(\bar{\partial}\varphi)$ has the SEP. All terms in (6.2) thus have the SEP and therefore (6.2) holds on X' , concluding the proof. \square

Proposition 6.2. *Let X be a reduced complex space of pure dimension n . Then $\bar{\partial}: \mathcal{A}_X^{p,q} \rightarrow \mathcal{A}_X^{p,q+1}$ and $0 \rightarrow \Omega_X^p \hookrightarrow \mathcal{A}_X^{p,0} \xrightarrow{\bar{\partial}} \mathcal{A}_X^{p,1} \xrightarrow{\bar{\partial}} \dots \xrightarrow{\bar{\partial}} \mathcal{A}_X^{p,n} \rightarrow 0$ is exact.*

Proof. Let φ be a $\bar{\partial}$ -closed section of $\mathcal{A}_X^{p,q}$ over some small neighborhood U of a given point $x \in X$; we may assume that U is an analytic subset of some pseudoconvex domain in some \mathbb{C}^N . As in the proof of Theorem 1.3 above one shows that, for suitable operators \mathcal{K} and \mathcal{P} , $\varphi = \bar{\partial}\mathcal{K}\varphi$ if $q \geq 1$ and $\varphi = \mathcal{P}\varphi$ is a section of Ω_X^p if $q = 0$.

It remains to see that $\bar{\partial}: \mathcal{A}_X^{p,q} \rightarrow \mathcal{A}_X^{p,q+1}$. Let φ be a $\bar{\partial}$ -closed section of $\mathcal{A}_X^{p,q}$ over some small neighborhood U of a given point $x \in X$; we may assume that φ is of the form (6.1) and we will use induction over ν . If $\nu = 0$ then $\varphi = \xi_0$ is smooth and so $\bar{\partial}\varphi$ is in $\mathcal{E}_X^{p,q+1} \subset \mathcal{A}_X^{p,q+1}$. Assume that $\bar{\partial}\varphi'$ is in $\mathcal{A}_X^{p,*}$ for φ' of the form (6.1) with $\nu = \ell - 1$. Since φ' is a section of $\mathcal{A}_X^{p,*}$ it follows from Proposition 5.3 that

$$(6.3) \quad \varphi' = \bar{\partial}\mathcal{K}_\ell\varphi' + \mathcal{K}_\ell(\bar{\partial}\varphi') + \mathcal{P}_\ell\varphi'$$

as currents on U'_{reg} for some sufficiently small neighborhood U' of x , cf. the proof of Theorem 1.3 above. As in that same proof (6.3) extends to hold on U' . The left hand side as well as the last term on the right hand side of (6.3) are obviously in $\mathcal{A}_X^{p,*}$ and since $\bar{\partial}\varphi'$ is in $\mathcal{A}_X^{p,*}$ by assumption and \mathcal{K} -operators preserve $\mathcal{A}_X^{p,*}$ also the second term on the right hand side is in $\mathcal{A}_X^{p,*}$. Hence, $\bar{\partial}\mathcal{K}_\ell\varphi'$ is a section of $\mathcal{A}_X^{p,*}$ over U' showing that $\bar{\partial}\varphi$ is in $\mathcal{A}_X^{p,*}$ for φ of the form (6.1) with $\nu = \ell$. \square

Notice that Theorem 1.1 follows from Propositions 6.1 and 6.2.

Proof of Proposition 1.5. Assume that condition (i) of Proposition 1.5 holds. Then, in view of the last paragraph in Section 4, any holomorphic p -form on the regular part at least extends to a section of ω_X^p ; in particular, such forms are meromorphic. It is thus clear from Proposition 5.5 that $\Omega^p(U) \rightarrow \Omega^p(U_{reg})$ is surjective for any open $U \subset X$; the injectivity is obvious. We remark that the implication (i) \Rightarrow (ii) also follows from [43, Satz III].

Assume that condition (ii) of Proposition 1.5 holds. In view of [44, Theorem 1.14, (d) \Rightarrow (b)] it is sufficient to show that the restriction map $H^1(U, \Omega_X^p) \rightarrow H^1(U_{reg}, \Omega_X^p)$ is injective for any open $U \subset X$. By Corollary 1.2, $H^1(U, \Omega_X^p) \simeq H^1(\mathcal{A}^{p,\bullet}(U), \bar{\partial})$, so let $\varphi \in \mathcal{A}^{p,\bullet}(U)$ be $\bar{\partial}$ -closed and assume that its image in $H^1(\mathcal{A}^{p,\bullet}(U_{reg}), \bar{\partial})$ vanishes, i.e., that there is $\psi \in \mathcal{A}^{p,0}(U_{reg})$ such that $\varphi = \bar{\partial}\psi$ on U_{reg} . Let $x \in U_{sing}$. By Theorem 1.1, there is a neighborhood $V \subset U$ of x and a $\psi' \in \mathcal{A}^{p,0}(V)$ such that $\varphi = \bar{\partial}\psi'$ in V . Then $\psi - \psi'$ is holomorphic on V_{reg} and so, by condition (ii), $\psi - \psi' \in \Omega^p(V)$. Hence, $\psi = \psi' + \psi - \psi'$ can be locally extended across U_{sing} to a section of $\mathcal{A}_X^{p,0}$. In view of the SEP, extensions are unique and so $\psi \in \mathcal{A}^{p,0}(U)$ and consequently $\bar{\partial}\psi \in \mathcal{A}^{p,1}(U)$. The equality $\varphi = \bar{\partial}\psi$ on U_{reg} therefore extends to hold on U by the SEP and so φ defines the zero element in $H^1(U, \Omega_X^p)$. \square

We conclude this subsection by showing that if condition (i) of Proposition 1.5 (or equivalently, condition (ii)) is satisfied and one *a priori* knows that X is locally a complete intersection, then X is smooth.

Assume that $X = \{f_1 = \cdots = f_\kappa = 0\} \subset D \subset \mathbb{C}^N$ has codimension κ and that $df_1 \wedge \cdots \wedge df_\kappa \neq 0$ on X_{reg} . Let $\tilde{\omega}$ be a meromorphic n -form in D such that the polar set of $\tilde{\omega}$ intersects X properly and such that, outside of the polar set of $\tilde{\omega}$, $df_1 \wedge \cdots \wedge df_\kappa \wedge \tilde{\omega} = dz$ for some local coordinates z in D . Let ω be the pullback of $\tilde{\omega}$ to X . Then ω is a holomorphic n -form on X_{reg} that is uniquely determined by dz and X ; in fact, ω is the Poincaré-Leray residue of the meromorphic form $dz/(f_1 \cdots f_\kappa)$. If ω has a strongly holomorphic extension to X , then, since $df_1 \wedge \cdots \wedge df_\kappa \wedge \omega = dz$, it follows that $df_1 \wedge \cdots \wedge df_\kappa \neq 0$ on X . Some *a priori* assumption is necessary since if $X = \{z_1 = z_4 = 0\} \cup \{z_2 = z_3 = 0\} \subset \mathbb{C}^4$ then one can check that any holomorphic 2-form on X_{reg} extends across X_{sing} to a section of Ω_X^2 .

6.2. The sheaves $\mathcal{B}_X^{n-p,\bullet}$. To define $\mathcal{B}_X^{n-p,q}$ we follow [41, Definition 4.1] and we say that a $(n-p, q)$ -current ψ on an open subset $U \subset X$ is a section of $\mathcal{B}_X^{n-p,q}$ over U if for every $x \in U$ the germ ψ_x can be written as a finite sum of terms

$$(6.4) \quad \xi_\nu \wedge \check{\mathcal{K}}_\nu(\cdots \xi_2 \wedge \check{\mathcal{K}}_2(\xi_1 \wedge \check{\mathcal{K}}_1(\omega \wedge \xi_0)) \cdots),$$

where ω is an $n-p$ -structure form and the ξ_j are smooth $(0, q)$ -forms with support where $\zeta \mapsto k_j(\zeta, z)$ is defined. Recall that ω is a $(n-p, *)$ -current with values in a bundle $\bigoplus_k E_k \restriction X$ so we need ξ_0 to take values in $\bigoplus_k E_k^* \restriction X$ to make $\omega \wedge \xi_0$ scalar-valued.

It is immediate from the definition and from the general properties of the $\check{\mathcal{K}}$ -operators that $\mathcal{B}_X^{n-p,q} \subset \mathcal{W}_X^{n-p,q}$, that $\mathcal{B}_{X_{reg}}^{n-p,q} = \mathcal{E}_{X_{reg}}^{n-p,q}$, that the $\check{\mathcal{K}}$ -operators and $\check{\mathcal{P}}$ -operators preserve $\bigoplus_q \mathcal{B}_X^{n-p,q}$, and that $\bigoplus_q \mathcal{B}_X^{n-p,q}$ is a module over $\bigoplus_q \mathcal{E}_X^{0,q}$. Let ψ be a smooth $(n-p, q)$ -form and let ω be an $n-p$ -structure form in a neighborhood of some point in X . Then, by Lemma 3.5, there is a smooth $(0, q)$ -form ψ' (with values in the appropriate bundle) such that $\psi = \omega_0 \wedge \psi'$. Hence we see that $\mathcal{E}_X^{n-p,q} \subset \mathcal{B}_X^{n-p,q}$. Let us also notice that if ψ is in $\mathcal{B}_X^{n-p,q}$ then $\bar{\partial}\psi$ has the SEP. In fact, we may assume that ψ is of the form (6.4) so that $\psi = \pi_* T \wedge \xi$, where T is given by (5.14), ξ is a smooth form, and π is the natural projection $X^{\nu+1} \rightarrow X_{z_0}$. Letting $h = h(z^0)$ be a generically non-vanishing holomorphic tuple on X , we have that $\bar{\partial}\chi(|h|^2/\epsilon) \wedge T \wedge \xi \rightarrow 0$ by Lemma 5.2. Hence, by Lemma 2.2, we see that $\bar{\partial}\psi$ has the SEP.

Proof of Theorem 1.7. We first interchange the roles of p and $n-p$ in the formulation of Theorem 1.7. Let $D'' \Subset D$ be a strictly pseudoconvex neighborhood of \overline{D}' and carry out the construction of $k(\zeta, z)$ and $p(\zeta, z)$ in Section 5 in $D'' \times D''$ using a weight $g(\zeta, z)$ with respect to $\zeta \in D'$ such that $\zeta \mapsto g(\zeta, z)$ is holomorphic in D' and $z \mapsto g(\zeta, z)$ has compact support in D'' . Let $\psi \in \mathcal{B}^{n-p,q}(X)$. By Proposition 5.6 we have

$$(6.5) \quad \psi = \bar{\partial}\check{\mathcal{K}}\psi + \check{\mathcal{K}}(\bar{\partial}\psi) + \check{\mathcal{P}}\psi$$

as currents on X'_{reg} . From what we noticed just before the proof all terms have the SEP and so (6.5) holds on X' . Notice that $\check{\mathcal{P}}\psi = A_q(\zeta) \wedge \omega_q(\zeta)$, where A_q is holomorphic. Since, if Ω_X^p is Cohen-Macaulay we may choose $\omega = \omega_0$ to be $\bar{\partial}$ -closed it follows that $\check{\mathcal{P}}\psi \in \omega^{n-p}(X')$ is $q = 0$ and $\check{\mathcal{P}}\psi = 0$ if $q \geq 1$. \square

Proof of Theorem 1.6. As in the proof above we interchange the roles of p and $n-p$ in the formulation of Theorem 1.6. We have already noted that (i) and (ii) hold.

To show that $\bar{\partial}: \mathcal{B}_X^{n-p,q} \rightarrow \mathcal{B}_X^{n-p,q+1}$ let ψ be a section of $\mathcal{B}_X^{n-p,q}$ in a neighborhood of some $x \in X$; we may assume that ψ is of the form (6.4) and we use induction over ν . If $\nu = 0$ then $\psi = \omega \wedge \xi_0$ and it is enough to see that $\bar{\partial}\omega$ is a section of $\mathcal{B}_X^{n-p,*}$ (with values in $E|_X$); but since $\bar{\partial}\omega = f\omega$ this is clear. The induction step is done in the same way as in the proof of Proposition 6.2.

To show that $\omega_X^{n-p,q}$ is coherent and that $\omega_X^{n-p} = \omega_X^{n-p,0}$ assume that X can be identified with an analytic subset of a strictly pseudoconvex domain $D \subset \mathbb{C}^N$. Recall that (2.6) is a resolution of Ω_X^p in D . Taking $\mathcal{H}om$ into Ω^N we get a complex isomorphic to $(\mathcal{O}(E_\bullet^*) \otimes \Omega^N, \bar{\partial})$ with associated cohomology sheaves isomorphic to $\mathcal{Ext}^\bullet(\Omega_X^p, \Omega^N)$, which are coherent; cf. Section 4. We define the map

$$\varrho_q: \mathcal{O}(E_{\kappa+q}^*) \otimes \Omega^N \rightarrow \mathcal{B}_X^{n-p,q}, \quad \varrho_q(\xi dz) = i^* \xi \cdot \omega_q.$$

Since

$$\begin{aligned} \bar{\partial}\varrho_q(\xi dz) &= i^* \xi \cdot \bar{\partial}\omega_q = i^* \xi \cdot f_{\kappa+q+1}|_X \omega_{q+1} = i^* f_{\kappa+q+1}^*|_X \xi \cdot \omega_{q+1} \\ &= \varrho_{q+1}(f_{\kappa+q+1}^* \xi dz) \end{aligned}$$

the map ϱ_\bullet is a map of complexes and so induces a map on cohomology. In view of Proposition 4.1 the proof will be complete if we show that ϱ_\bullet is a quasi-isomorphism.

Since $i_* \omega_q = R_{\kappa+q} \wedge dz$ it follows from [6, Theorem 7.1] that the map on cohomology is injective. For the surjectivity, let $\psi \in \mathcal{B}^{n-p,q}(X)$ be $\bar{\partial}$ -closed and choose a weight $g(\zeta, z)$ in the kernels $k(\zeta)$ and $p(\zeta, z)$ with respect to ζ in some $D' \Subset D$ such that $\zeta \mapsto g(\zeta, z)$ is holomorphic in D' and $z \mapsto g(\zeta, z)$ has compact support in D . As in the proof of Theorem 1.7 we get that $\psi = \bar{\partial}\check{\mathcal{K}}\psi + \check{\mathcal{P}}\psi$ on $X'_{reg} := X_{reg} \cap D'$ and so the cohomology class of ψ is represented by $\check{\mathcal{P}}\psi$. From the definition of $p(\zeta, z)$ in Section 5 we see that

$$\check{\mathcal{P}}\psi(\zeta) = \pm \omega_q(\zeta) \wedge \int_{X_z} \tilde{p}_{\kappa+q}(\zeta, z) \wedge \psi(z)$$

and $\zeta \mapsto \tilde{p}_{\kappa+q}(\zeta, z)$ is a section of $\mathcal{O}(E_{\kappa+q}^*)$ over D' by the choice of g . We finally show that

$$(6.6) \quad f_{\kappa+q+1}^* \int_{X_z} \tilde{p}_{\kappa+q}(\zeta, z) \wedge \psi(z) = 0.$$

First notice that it follows from (5.6) and (5.7) that, for each k , $\tilde{p}_k(\zeta, z) \wedge d\eta = H_k^0 \wedge g_{N-k}$. Moreover,

$$\begin{aligned} f_{k+1}^* H_k^0 \wedge g_{N-k} &= H_k^0 f_{k+1} \wedge g_{N-k} = (f_1(z) H_{k+1}^1 + \delta_\eta H_{k+1}^0) \wedge g_{N-k} \\ &= f_1(z) H_{k+1}^1 \wedge g_{N-k} \pm H_{k+1}^0 \wedge \delta_\eta g_{N-k} \\ &= f_1(z) H_{k+1}^1 \wedge g_{N-k} \pm H_{k+1}^0 \wedge \bar{\partial} g_{N-k-1} \\ &= f_1(z) H_{k+1}^1 \wedge g_{N-k} + \bar{\partial}(H_{k+1}^0 \wedge g_{N-k-1}) \\ &=: (f_1(z) A_k + \bar{\partial} B_k) \wedge d\eta, \end{aligned}$$

where A_k and B_k take values in $\text{Hom}(E_{k+1}^\zeta, E_1^z)$ and $\text{Hom}(E_{k+1}^\zeta, E_0^z)$ respectively; the second equality follows from the properties of the Hefer morphisms, the third by noting that $0 = \delta_\eta(H_{k+1}^0 \wedge g_{N-k}) = \delta_\eta H_{k+1}^0 \wedge g_{N-k} \pm H_{k+1}^0 \wedge \delta_\eta g_{N-k}$, the forth since g is a weight, the fifth since the Hefer morphisms are holomorphic, and the sixth by collecting all $d\eta_j$. Hence, we get that $f_{k+1}^* \tilde{p}_k(\zeta, z) = f_1(z) A_k + \bar{\partial} B_k$. Since $f_1|_X = 0$ and by Stokes' theorem, (6.6) follows. \square

7. SERRE DUALITY

7.1. The trace map. The key to define the trace map is the following slight generalization of [41, Theorem 5.1]; the proof of that theorem goes through in our case essentially verbatim. Notice however that in *ibid.* the notation $\mathcal{A}_X^{n,*}$ is used in place of our $\mathcal{B}_X^{n,*}$.

Theorem 7.1. *Let X be a reduced complex space of pure dimension n . There is a unique map*

$$\wedge: \mathcal{A}_X^{p,q} \times \mathcal{B}_X^{n-p,q'} \rightarrow \mathcal{W}_X^{n,q+q'}$$

extending the exterior product on X_{reg} . Moreover, if φ and ψ are sections of $\mathcal{A}_X^{p,q}$ and $\mathcal{B}_X^{n-p,q'}$, respectively, then $\bar{\partial}(\varphi \wedge \psi)$ has the SEP.

Let $\varphi \in \mathcal{A}^{p,q}(X)$ and $\psi \in \mathcal{B}^{n-p,n-q}(X)$ and assume that at least one of φ and ψ has compact support. By Theorem 7.1, $\varphi \wedge \psi$ is a well-defined section of $\mathcal{W}_X^{n,n}$ with compact support and we define our trace map on the level of currents by mapping (φ, ψ) to the action of $\varphi \wedge \psi$ on the constant function 1 on X ; if h is a generically non-vanishing holomorphic section of a Hermitian vector bundle such that $\{h = 0\} \supset X_{sing}$ this may be computed as $(\varphi, \psi) \mapsto \lim_{\epsilon \rightarrow 0} \int_X \chi(|h|^2/\epsilon) \varphi \wedge \psi$. This trace map on the level of currents induces a trace map on the level of cohomology. In fact, assume that φ and ψ are $\bar{\partial}$ -closed and that one of them, say φ , is $\bar{\partial}$ -exact so that $\varphi = \bar{\partial}\tilde{\varphi}$ for some $\tilde{\varphi} \in \mathcal{A}^{p,q-1}(X)$ with compact support if φ has. Then $\varphi \wedge \psi = \bar{\partial}(\tilde{\varphi} \wedge \psi)$ at least on X_{reg} . However, by Theorem 7.1, both the left and the right hand side has the SEP so this holds on X . Hence, (φ, ψ) is mapped to 0 by Stokes' theorem.

7.2. Local duality. Let \tilde{X} be an analytic subset of $\overline{D} \subset \mathbb{C}^N$, where D is pseudoconvex, and set $X := \tilde{X} \cap D$. Let F be a holomorphic vector bundle on X and let \mathcal{F} be the associated locally free \mathcal{O}_X -module. Since X is Stein and $\mathcal{F} \otimes \Omega_X^p$ is coherent it follows from Corollary 1.2 that the complex

$$0 \rightarrow \mathcal{A}^{p,0}(X, F) \xrightarrow{\bar{\partial}} \mathcal{A}^{p,1}(X, F) \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} \mathcal{A}^{p,n}(X, F) \rightarrow 0$$

is exact except for on the level 0 where the cohomology is $\Omega^p(X, F)$. We endow $\Omega^p(X, F)$ with the standard canonical Fréchet space topology, see, e.g., [18, Chapter IX].

Theorem 7.2. *Let $\mathcal{B}_c^{n-p,q}(X, F^*)$ be the space of sections of $\mathcal{F}^* \otimes \mathcal{B}_X^{n-p,q}$ with compact support in X . The complex*

$$(7.1) \quad 0 \rightarrow \mathcal{B}_c^{n-p,0}(X, F^*) \xrightarrow{\bar{\partial}} \mathcal{B}_c^{n-p,1}(X, F^*) \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} \mathcal{B}_c^{n-p,n}(X, F^*) \rightarrow 0$$

is exact except for on the level n and the pairing

$$(7.2) \quad \Omega^p(X, F) \times H^n(\mathcal{B}_c^{n-p,\bullet}(X, F^*), \bar{\partial}) \rightarrow \mathbb{C}, \quad (\varphi, [\psi]) \mapsto \int_X \varphi \cdot \psi$$

makes $H^n(\mathcal{B}_c^{n-p,\bullet}(X, F^), \bar{\partial})$ the topological dual of $\Omega^p(X, F)$.*

Sketch of proof. Since we are in the local situation we may assume that an element in $\mathcal{B}_c^{n-p,q}(X, F^*)$ is just a tuple of elements in $\mathcal{B}_c^{n-p,q}(X)$ and carry out the following argument component-wise. Let $\psi \in \mathcal{B}_c^{n-p,q}(X)$ be $\bar{\partial}$ -closed. Let $D' \Subset D'' \subset D$, where $\text{supp } \psi \subset D'$ and D'' is strictly pseudoconvex, and construct $k(\zeta, z)$ and $p(\zeta, z)$ as in Section 5 with a weight $g(\zeta, z)$ with respect to $z \in D'$ such that $z \mapsto g(\zeta, z)$ is holomorphic in D' and $\zeta \mapsto g(\zeta, z)$ has compact support in D'' . Then $p(\zeta, z) =$

$\sum_k \tilde{p}_{\kappa+k}(\zeta, z) \wedge \omega_k(\zeta)$, where $\zeta \mapsto \tilde{p}_{\kappa+k}(\zeta, z)$ has compact support in D'' and $z \mapsto \tilde{p}_{\kappa+k}(\zeta, z)$ is a section of Ω_X^p over $X' := X \cap D'$.

As in the proof of Theorem 1.7 we get $\psi = \bar{\partial} \check{\mathcal{K}}\psi + \check{\mathcal{P}}\psi$ in X' . From the properties of $p(\zeta, z)$ we get that $\check{\mathcal{P}}\psi = 0$ if $q < n$ so (7.1) is exact except for on the level n . If $q = n$ then the cohomology class of ψ is represented by $\check{\mathcal{P}}\psi$ and

$$\check{\mathcal{P}}\psi = \pm \sum_{k \geq 0} \omega_k(\zeta) \wedge \int_{X_z} \tilde{p}_{\kappa+k}(\zeta, z) \wedge \psi(z).$$

Hence, if $\int_X \varphi \psi = 0$ for all $\varphi \in \Omega^p(X)$ then $\check{\mathcal{P}}\psi = 0$ and the cohomology class of ψ thus is 0. It follows that $H^n(\mathcal{B}_c^{n-p,\bullet}(X), \bar{\partial})$, via (7.2), is a subset of the topological dual of $\Omega^p(X)$.

Let λ be a continuous linear functional on $\Omega^p(X)$. Then λ induces a continuous functional $\tilde{\lambda}$ on $\Omega^p(D)$ that has to be carried by some compact $K \Subset D$. By the Hahn-Banach theorem there is an $(N-p, N)$ -current μ of order 0 in D with support in a neighborhood $U(K) \Subset D$ of K such that $\tilde{\lambda}(\tilde{f}) = \int \tilde{f} \wedge \mu$ for all $\tilde{f} \in \Omega^p(D)$. Now choose a weight $g(\zeta, z)$ with respect to $z \in U(K)$ that is holomorphic for $z \in U(K)$ and has compact support in D_ζ and let $p(\zeta, z) = \sum_k \tilde{p}_{\kappa+k}(\zeta, z) \wedge \omega_k(\zeta)$ be a corresponding integral kernel. We set

$$\check{\mathcal{P}}\mu := \sum_{k \geq 0} \omega_k(\zeta) \wedge \int_{D_z} \tilde{p}_{\kappa+k}(\zeta, z) \wedge \mu(z)$$

and observe that $\check{\mathcal{P}}\mu \in \mathcal{B}_c^{n-p,n}(X)$. Let $\varphi \in \Omega^p(X)$ and set $\tilde{\varphi} := \check{\mathcal{P}}\varphi$. Then $\tilde{\varphi} \in \Omega^p(U(K))$ by the choice of weight and moreover, $\tilde{\varphi}|_{U(K) \cap X} = \varphi|_{U(K) \cap X}$. We get

$$\lambda(\varphi) = \tilde{\lambda}(\tilde{\varphi}) = \int_{D_z} \tilde{\varphi} \wedge \mu = \int_{D_z} \check{\mathcal{P}}\varphi \wedge \mu = \int_{X_\zeta} \varphi \wedge \check{\mathcal{P}}\mu$$

and so λ is given by integration against $\check{\mathcal{P}}\mu \in \mathcal{B}_c^{n-p,n}(X)$. For more details of the last part of the proof see the proof of [41, Theorem 6.1]. \square

7.3. Global duality. Let us briefly recall how one can patch up the local duality to the global one of Theorem 1.8; cf., e.g., [41, Section 6.2]. Let $\mathcal{U} := \{U_j\}$ be a locally finite open covering of X such that each U_j can be identified with an analytic subset of some pseudoconvex domain in some \mathbb{C}^N . In view of Theorem 1.1 and Corollary 1.2 this gives us a Leray covering for $\mathcal{F} \otimes \Omega_X^p$. Recall that spaces of sections of $\mathcal{F} \otimes \Omega_X^p$ has a standard Fréchet space structure. We let $C^k(\mathcal{U}, \mathcal{F} \otimes \Omega_X^p)$ be the group of formal sums

$$\sum_{i_0 \cdots i_k} \varphi_{i_0 \cdots i_k} U_{i_0} \wedge \cdots \wedge U_{i_k}, \quad \varphi_{i_0 \cdots i_k} \in \mathcal{F} \otimes \Omega^p(U_{i_0} \cap \cdots \cap U_{i_k}),$$

with the product topology and the suggestive computation rules. Each element of $C^k(\mathcal{U}, \mathcal{F} \otimes \Omega_X^p)$ thus has a unique representation of the form $\sum_{i_0 < \cdots < i_k} \varphi_{i_0 \cdots i_k} U_{i_0} \wedge \cdots \wedge U_{i_k}$ that we will abbreviate as $\sum'_{|I|=k+1} \varphi_I U_I$. We define a coboundary operator $\delta: C^k(\mathcal{U}, \mathcal{F} \otimes \Omega_X^p) \rightarrow C^{k+1}(\mathcal{U}, \mathcal{F} \otimes \Omega_X^p)$ by

$$\delta \sum'_{|I|=k+1} \varphi_I U_I := \sum'_{|I|=k+1} \varphi_I U_I \wedge \sum_j U_j = \sum'_{|I|=k+1} \sum_j \varphi_I \upharpoonright_{U_I \cap U_j} U_I \wedge U_j,$$

which is continuous, and we get the following complex of Fréchet spaces

$$(7.3) \quad 0 \rightarrow C^0(\mathcal{U}, \mathcal{F} \otimes \Omega_X^p) \xrightarrow{\delta} C^1(\mathcal{U}, \mathcal{F} \otimes \Omega_X^p) \xrightarrow{\delta} \cdots.$$

The q^{th} cohomology group of this complex is isomorphic to $H^q(X, \mathcal{F} \otimes \Omega_X^p)$ and in fact, the standard topology on $H^q(X, \mathcal{F} \otimes \Omega_X^p)$ is defined so that the isomorphism also is a homeomorphism.

Let B^{n-p} be the presheaf (see, e.g., [12, Section 3]) defined by assigning to each open $U \subset X$ the space

$$B^{n-p}(U) := H^n(\mathcal{B}_c^{n-p,\bullet}(U, F^*), \bar{\partial})$$

and for $U' \subset U$ the inclusion map $i_U^{U'}: B^{n-p}(U') \rightarrow B^{n-p}(U)$ given by extension by 0. We let, for $k \geq 0$, $C_c^{-k}(\mathcal{U}, B^{n-p})$ be the group of formal sums

$$\sum_{i_0 \dots i_k} [\psi_{i_0 \dots i_k}]_{\bar{\partial}} U_{i_0}^* \wedge \cdots \wedge U_{i_k}^*, \quad \psi_{i_0 \dots i_k} \in \mathcal{B}_c^{n-p,n}(U_{i_0} \cap \cdots \cap U_{i_k}, F^*),$$

with the suggestive computation properties and only finitely many $[\psi_{i_0 \dots i_k}]_{\bar{\partial}}$ non-zero. We define the coboundary operator $\delta^*: C_c^{-k}(\mathcal{U}, B^{n-p}) \rightarrow C_c^{-k+1}(\mathcal{U}, B^{n-p})$ by

$$\delta^* \sum'_{|I|=k+1} [\psi_I] U_I^* := \sum_j U_j \lrcorner \sum'_{|I|=k+1} [\psi_I] U_I^* = \sum'_{|I|=k+1} \sum_j i_{U_I \setminus \{j\}}^{U_I} [\psi_I] U_j \lrcorner U_I^*,$$

and we get the complex

$$(7.4) \quad 0 \leftarrow C_c^0(\mathcal{U}, B^{n-p}) \xleftarrow{\delta^*} C_c^{-1}(\mathcal{U}, B^{n-p}) \xleftarrow{\delta^*} \cdots.$$

By Theorem 7.2, $C_c^{-k}(\mathcal{U}, B^{n-p})$ is the topological dual of $C^k(\mathcal{U}, \mathcal{F} \otimes \Omega_X^p)$ via the pairing $C^k(\mathcal{U}, \mathcal{F} \otimes \Omega_X^p) \times C_c^{-k}(\mathcal{U}, B^{n-p}) \rightarrow \mathbb{C}$ given by

$$(7.5) \quad (\varphi, [\psi]_{\bar{\partial}}) = \left(\sum'_{|I|=k+1} \varphi_I U_I, \sum'_{|I|=k+1} [\psi_I]_{\bar{\partial}} U_I^* \right) \mapsto \int_X \varphi \lrcorner \psi = \sum'_{|I|=k+1} \int_X \varphi_I \wedge \psi_I.$$

Moreover, if $\varphi \in C^{k-1}(\mathcal{U}, \mathcal{F} \otimes \Omega_X^p)$ and $[\psi] \in C_c^{-k}(\mathcal{U}, B^{n-p})$, then

$$\int_X \varphi \lrcorner \delta^* \psi = \int_X \varphi \lrcorner \left(\sum_j U_j \lrcorner \psi \right) = \int_X (\varphi \wedge \sum_j U_j) \lrcorner \psi = \int_X \delta \varphi \lrcorner \psi$$

and so (7.4) is the dual complex of (7.3). It follows, see, e.g., [39, Lemme 2], that

$$(7.6) \quad \text{Ker}(\delta^*: C_c^{-q}(\mathcal{U}, B^{n-p}) \rightarrow C_c^{-q+1}(\mathcal{U}, B^{n-p})) / \overline{\delta^* C_c^{-q-1}(\mathcal{U}, B^{n-p})}$$

is the topological dual of

$$(7.7) \quad \text{Ker}(\delta: C^q(\mathcal{U}, \mathcal{F} \otimes \Omega_X^p) \rightarrow C^{q+1}(\mathcal{U}, \mathcal{F} \otimes \Omega_X^p)) / \overline{\delta C^{q-1}(\mathcal{U}, \mathcal{F} \otimes \Omega_X^p)}.$$

Now, if $H^q(X, \mathcal{F} \otimes \Omega_X^p)$ and $H^{q+1}(X, \mathcal{F} \otimes \Omega_X^p)$ are Hausdorff, then the closure signs in (7.6) and (7.7) are superfluous and so $H^{-q}(C_c^\bullet(\mathcal{U}, B^{n-p}), \delta^*)$ is the topological dual of $H^q(X, \mathcal{F} \otimes \Omega_X^p)$ in this case, via the pairing induced by (7.5).

To understand $H^{-q}(C_c^\bullet(\mathcal{U}, B^{n-p}), \delta^*)$, consider the double complex

$$K^{-i,j} := C_c^{-i}(\mathcal{U}, \mathcal{B}_c^{n-p,j}),$$

where $\mathcal{B}_c^{n-p,j}$ is the presheaf $U \mapsto \mathcal{B}_c^{n-p,j}(U)$ with inclusion maps given by extending by 0, the map $K^{-i,j} \rightarrow K^{-i+1,j}$ is δ^* , and the map $K^{-i,j} \rightarrow K^{-i,j+1}$ is $\bar{\partial}$. For each $i \geq 0$ the “row” $K^{-i,\bullet}$ is, by Theorem 7.2, exact except for on the level

n where the cohomology is $C_c^{-i}(\mathcal{U}, B^{n-p})$. Since the \mathcal{B}_X -sheaves are fine it follows from, e.g., [41, Lemma 6.2] that, for each $j \geq 0$, the “column” $K^{\bullet,j}$ is exact except for on the level 0 where the cohomology is $\mathcal{B}_c^{n-p,j}(X, F^*)$. From, e.g., a spectral sequence argument it thus follows that

$$(7.8) \quad H^{-q}(C_c^\bullet(\mathcal{U}, B^{n-p}), \delta^*) \simeq H^{n-q}(\mathcal{B}_c^{n-p,\bullet}(X, F^*), \bar{\partial}).$$

Hence, we have a non-degenerate pairing (1.2) but we have not proved that it is given by (1.3). To do this one makes the isomorphisms $H^q(\mathcal{A}^{p,\bullet}(X, F), \bar{\partial}) \simeq H^q(X, \mathcal{F} \otimes \Omega_X^p)$ and (7.8) explicit; see the proof of [41, Theorem 1.3] for details.

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